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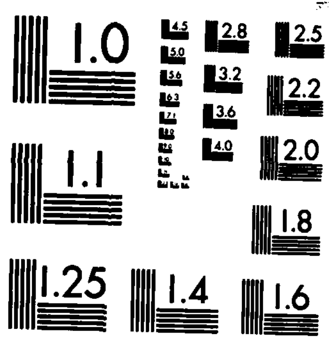

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BIAXIAL BUCKLING OF SPECIALLY
ORTHOTROPIC, SYMMETRIC
COMPOSITE RECTANGULAR PLATES

THESIS

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SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

AFIT/GAE/AA/84D-15

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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

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Master of Science in Aeronautical Engineering

James P. McFadden, B.S.

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Preface

My basic interest and job-related experience in the analysis of composite structures compelled me to explore this wide area for a research topic. I also possessed a working understanding of the theory of isotropic uniaxial plate buckling theory, so I merged these two concepts into a challenging task.

The main individual who provided considerable guidance and patience throughout this project is my thesis advisor, Dr. E. J. Brunelle. Without his help and indeed his initial idea and groundwork, this report would never have reached a satisfactory conclusion.

James P. McFadden

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and m increases as a_o/b_o increases.

Again, transition points represent quantities of special interest. First consideration is given to the determination of the point where the K_{x_o} versus a_o/b_o curve for (m=1,(n+1)) intersects the K_{x_o} versus a_o/b_o curve for (m=1,n). Both curves utilize a fixed value of K_{y_o}. Equating the right-hand side of equation (20) for (m=1,(n+1)) to an equivalent right-hand side of equation (20) for (m=1,n) determines the value of a_o/b_o at the transition point.

$$-K_{y_o}(n+1)^2 + 1.0/(a_o/b_o)^2_{tr_2} + (n+1)^4 (a_o/b_o)^2_{tr_2} = -K_{y_o}(n)^2 + 1.0/(a_o/b_o)^2_{tr_2} + (n)^4 (a_o/b_o)^2_{tr_2} \quad (27)$$

$$(a_o/b_o)_{tr_2} = \{K_{y_o}[(n+1)^2 - n^2]/[(n+1)^4 - n^4]\}^{1/2} = \{K_{y_o}/[(n+1)^2 + n^2]\}^{1/2} \quad (28)$$

where

$(a_o/b_o)_{tr_2}$ = value of characteristic plate aspect ratio at transition from (m=1,(n+1)) to (m=1,n)

Equation (28) gives the value of a_o/b_o for any fixed K_{y_o} greater than zero for the intersection of the K_{x_o} versus a_o/b_o curve for (m=1,(n+1)) with the K_{x_o} versus a_o/b_o curve for (m=1,n). Furthermore, substitution of the quantity $(a_o/b_o)_{tr_2}$ into equation (20) returns K_{x_o} for the transitional value of a_o/b_o.

Of course since $(a_o/b_o)_{tr_1}$ and $K_{x_{o_{tr_1}}}$ signify coordinates of a mutual point of the m and (m+1) curves, either equation presented in equation (26) yields the correct answer.

Table I gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets based upon equations (21), (25), and (26). The values of m and n (=1) are clearly marked for each point, and each entry which corresponds to a transition point is superscripted with a star (*). Note that K_{y_o} is less than or equal to zero for all points because compressive K_{y_o} will be considered separately in the next section.

Two key observations can be made after review of the data in Table I. First, the transition values of a_o/b_o increase as K_{y_o} becomes less tensile. This trend is most pronounced for the initial transition points, and its effect diminishes as a_o/b_o becomes large. Second, regardless of the tensile magnitude of K_{y_o}, K_{x_o} attains a limiting value of 2.0 as a_o/b_o approaches infinity. Therefore, K_{x_o}, for a tensile K_{y_o}, is effectively independent of K_{y_o} for large a_o/b_o.

Compression Applied in y_o-Direction

For K_{y_o} greater than zero, a value of n=1 no longer uniformly satisfies equation (20) for minimum K_{x_o}. Small a_o/b_o quantities in fact dictate that m achieve a value of unity and n increase as a_o/b_o decreases. As a_o/b_o becomes relatively large, however, the pattern demonstrated in the section for K_{y_o} less than or equal to zero reappears -- n=1

$$[(a_o/b_o)_{tr_1}]^4 \{(m+1)^2 - m^2\}/[m(m+1)]^2$$
$$-[(a_o/b_o)_{tr_1}]^2 K_{y_o} \{(m+1)^2 - m^2\}/[m(m+1)]^2$$
$$-[(m+1)^2 - m^2] = 0 \qquad (23)$$

where

$(a_o/b_o)_{tr_1}$ = value of characteristic plate aspect ratio at transition from m to (m+1)

Application of the quadratic equation fixes $((a_o/b_o)_{tr_1})$ Only one root is applicable since the second yields a negative aspect ratio.

$$((a_o/b_o)_{tr_1})^2 = 0.5 \{m(m+1)\}^2 \quad [K_{y_o}/(m(m+1))^2$$
$$+ \{(K_{y_o}^2/(m(m+1))^4 + 4.0/(m(m+1))^2\}^{1/2}] \qquad (24)$$

$$(a_o/b_o)_{tr_1} = (0.5)^{1/2} \{ K_{y_o} + [K_{y_o}^2$$
$$+ 4 (m(m+1))^2]^{1/2} \}^{1/2} \qquad (25)$$

Equation (25) gives the value of a_o/b_o, for any fixed K_{y_o} less than or equal to zero, for the intersection of the K_{x_o} versus a_o/b_o curve for m with the K_{x_o} versus a_o/b_o curve for (m+1). Furthermore, substitution of $(a_o/b_o)_{tr_1}$ into equation (21) returns K_{x_o} for the transitional value of a_o/b_o.

$$K_{x_{o_{tr_1}}} = -K_{y_o}/(m+1)^2 + [(m+1)/(a_o/b_o)_{tr_1}]^2$$
$$+ [(a_o/b_o)_{tr_1}/(m+1)]^2$$

-or- (26)

$$K_{x_{o_{tr_1}}} = -K_{y_o}/m^2 + [m/(a_o/b_o)_{tr_1}]^2$$
$$+ [(a_o/b_o)_{tr_1}/m]^2$$

where

$K_{x_{o_{tr_1}}}$ = x_o-buckling coefficient at transition from m to m+1

buckling.

Consider two cases of equation (20) -- the first for K_{y_0} less than or equal to zero and the second for positive K_{y_0}. In other words, the first case reflects buckling behavior in the presence of tension or no loading in the y_0-direction, and the second case illustrates such characteristics for compression in the transverse direction.

<u>Tension or Zero Loading Applied in the y_0-Direction</u>

For K_{y_0} less than or equal to zero, it is obvious that the minimum K_{x_0} in all situations corresponds to the lowest value of n (n=1). Thus, equation (20) becomes:

$$K_{x_0} = -K_{y_0}/m^2 + (mb_0/a_0)^2 + (a_0/mb_0)^2 \qquad (21)$$

For small values of the characteristic ratio a_0/b_0, m=1 produces the smallest K_{x_0}, yet as a_0/b_0 increases, m greater than one yields the minimum K_{x_0}. Remember that m is constrained to be an integer quantity. Of special interest, therefore, is the location of those transition points where the K_{x_0} versus a_0/b_0 curve for m intersects the K_{x_0} versus a_0/b_0 curve for (m+1). Both curves utilize a fixed value of K_{y_0}. Equating the right-hand side of equation (21) for m to an equivalent right-hand side of equation (21) for (m+1) determines the value of a_0/b_0 at the transition point.

$$\begin{aligned}-K_{y_0}/(m+1)^2 &+ [(m+1)/(a_0/b_0)_{tr_1}]^2 \\ &+ [(a_0/b_0)_{tr_1}/(m+1)]^2 \quad = \\ -K_{y_0}/m^2 &+ [m/(a_0/b_0)_{tr_1}]^2 \\ &+ [(a_0/b_0)_{tr_1}/m]^2 \qquad (22)\end{aligned}$$

$$w_{,x_ox_o} = -F(m\pi/a_o)^2 \sin(m\pi x_o/a_o) \sin(n\pi y_o/b_o)$$

$$w_{,x_ox_ox_ox_o} = F(m\pi/a_o)^4 \sin(m\pi x_o/a_o) \sin(n\pi y_o/b_o)$$

$$w_{,y_oy_o} = -F(n\pi/b_o)^2 \sin(m\pi x_o/a_o) \sin(n\pi y_o/b_o) \tag{17}$$

$$w_{,y_oy_oy_oy_o} = F(n\pi/b_o)^4 \sin(m\pi x_o/a_o) \sin(n\pi y_o/b_o)$$

Upon substitution of the relations in equation (17) into equation (14), the buckling equation produces the following sequence:

$$F \sin(m\pi x_o/a_o) \sin(n\pi y_o/b_o) \{(m\pi/a_o)^4 + (n\pi/b_o)^4 - K_{x_o}[(m\pi^2)/(a_ob_o)]^2 - K_{y_o}[(n\pi^2)/(a_ob_o)]^2\} = 0 \tag{18}$$

$$(m/a_o)^4 + (n/b_o)^4 - K_{x_o}[m/(a_ob_o)]^2 - K_{y_o}[n/(a_ob_o)]^2 = 0 \tag{19}$$

$$K_{x_o} = -K_{y_o}(n/m)^2 + (mb_o/a_o)^2 + [(a_on^2)/(mb_o)]^2 \tag{20}$$

As shown in equation (20), an exact solution for K_{x_o} exists for any choice of K_{y_o}. Of course, the minimum value of K_{x_o} returned from trials with various values of m and n is of the greatest significance. This minimum K_{x_o} represents the smallest level of the coefficient necessary to initiate

II. Flat Rectangular Composite Laminate Simply Supported on All Four Sides

The boundary conditions for a laminate simply supported on all sides specify that the vertical displacement along each edge and the normal component of moment to each edge must vanish in the affine space. In equation form, the following must hold:

$$
\begin{aligned}
&\text{on edge } x_o = 0\,, && w = 0 \;; && w_{,x_o x_o} = 0 \\
&\text{on edge } x_o = a_o\,, && w = 0 \;; && w_{,x_o x_o} = 0 \\
&\text{on edge } y_o = 0\,, && w = 0 \;; && w_{,y_o y_o} = 0 \\
&\text{on edge } y_o = b_o\,, && w = 0 \;; && w_{,y_o y_o} = 0
\end{aligned}
\qquad (15)
$$

The sine series shown below, which models the displacement w, satisfies each stipulation of equation (15) and will be applied to the general equation (14).

$$w = F \sin\,(m\pi\, x_o/a_o)\ \ \sin\,(n\pi\, y_o/b_o) \qquad (16)$$

where

F= arbitrary coefficient

m= integer which can take on any of the values 1,2,3, ..

n= integer which can take on any of the values 1,2,3, ..

Correct partial differentiation of equation (16) yields the following:

the range zero to one (1). Furthermore, the character of the solution is rather invariant of this ratio over the range zero to one. In particular, a zero value of the starred bending stiffness ratio constitutes a close approximation to plate buckling behavior (1). This null value will be employed throughout the report.

Application of all notational simplifications gives the final form of the biaxial buckling equation:

$$w_{,x_0x_0x_0x_0} + w_{,y_0y_0y_0y_0} + K_{x_0}(\pi/b_0)^2 \, w_{,x_0x_0} + K_{y_0}(\pi/a_0)^2 \, w_{,y_0y_0} = 0 \qquad (14)$$

straightforward manner:

$$
\begin{aligned}
w_{,x} &= (1/A)\, w_{,x_o} \\
w_{,y} &= (1/B)\, w_{,y_o} \\
w_{,z} &= w_{,z_o}
\end{aligned}
\tag{8}
$$

Combining equations (5),(6), and (8) yields the buckling equation expressed in the desired doubly affine space:

$$
\begin{aligned}
&(D_{11}/A^4)\, w_{,x_o x_o x_o x_o} \\
&+ 2(D_{12} + 2D_{66})/(AB)^2 \; w_{,x_o x_o y_o y_o} \\
&+ (D_{22}/B^4)\, w_{,y_o y_o y_o y_o} \quad - (N_x/A^2)\, w_{,x_o x_o} \\
&\qquad - (N_y/B^2)\, w_{,y_o y_o} = 0
\end{aligned}
\tag{9}
$$

For maximum simplicity equation (9) dictates the choice of the arbitrary constants, A and B. For coefficients of unity for the first and third terms $A = D_{11}^{1/4}$ and $B = D_{22}^{1/4}$ The buckling equation then becomes:

$$
\begin{aligned}
&w_{,x_o x_o x_o x_o} + 2(D_{12} + 2D_{66})/(D_{11}\, D_{22})^{1/2} \; w_{,x_o x_o y_o y_o} \\
&+ w_{,y_o y_o y_o y_o} - N_x/(D_{11})^{1/2} \; w_{,x_o x_o} \\
&\qquad - N_y/(D_{22})^{1/2} \; w_{,y_o y_o} = 0
\end{aligned}
\tag{10}
$$

Further notational simplifications are:

$$D^* = 2(D_{12} + 2D_{66})/(D_{11}\, D_{22})^{1/2} \tag{11}$$

$$-N_x/(D_{11})^{1/2} = K_{x_o}\, (\pi/b_o)^2 \tag{12}$$

$$-N_y/(D_{22})^{1/2} = K_{y_o}\, (\pi/a_o)^2 \tag{13}$$

where

$\pi = 3.1415927$

K_{x_o} = buckling coefficient in x_o-direction

K_{y_o} = buckling coefficient in y_o-direction

D^* = starred bending stiffness ratio

This starred bending stifiness ratio D^* must lie within

$$D_{26} = 0 \tag{4}$$

Differentiating the moment quantities given by equation (2) and substitution into equation (1) finally yields the buckling equation for a symmetric, specially orthotropic composite laminate expressed in Cartesian coordinates:

$$D_{11} w_{,xxxx} + 2(D_{12} + 2D_{66}) w_{,xxyy} + D_{22} w_{,yyyy} - N_x w_{,xx} - N_y w_{,yy} = 0 \tag{5}$$

The coordinate system for this equation is now transformed from Cartesian to doubly affine space (1). The equations relating the coordinates are straightforward:

$$\begin{aligned} Ax_o &= x \\ By_o &= y \\ z_o &= z \end{aligned} \tag{6}$$

where

x_o = longitudinal plate coordinate in affine space

y_o = transverse plate coordinate in affine space

z_o = z = vertical plate coordinate in affine/real space

A = arbitrary constant coefficient

B = arbitrary constant coefficient

In addition, if the rectangular plate measures a units in the longitudinal and b units in the transverse direction, the following must hold:

$$\begin{aligned} a_o &= a/A \\ b_o &= b/B \\ a_o/b_o &= (a/b)(B/A) \end{aligned} \tag{7}$$

From equations (6) differentiation in the affine space compares to differentiation in Cartesian space again in a

I. Derivation of Buckling Equation for Specially Orthotropic Laminated Plates, Symmetric about Middle Surface, Expressed in Doubly Affine Space Coordinates

The general buckling equation for a rectangular plate under biaxial loading is given by (2:245) :

$$M_{x,xx} + 2\,M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + N_y w_{,yy} = 0 \qquad (1)$$

where

M= moment per unit length

N= normal force per unit length positive in tension

w= displacement of middle surface of the laminate

The moments are related to the displacement field of a symmetric laminate by the matrix equation (1:149-156):

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \qquad (2)$$

where

D_{ij} = component of bending stiffness array

u = displacement in x-direction of plate

v = displacement in y-direction of plate

Now consideration is narrowed from that of a symmetric anisotropic laminate to one both specially orthotropic and symmetric about the middle surface. These restrictions cause the following terms in the stiffness array to vanish:

$$D_{16} = 0 \qquad (3)$$

Abstract

The biaxial plate buckling problem for specially orthotropic, symmetric laminates is transformed from Cartesian to doubly affine space. Setting the starred bending stiffness ratio D^* (which ranges from zero to one) to the null value enables ready and very accurate solution of the buckling problem. Seven sets of boundary restraint configurations are examined, and corresponding buckling surfaces are presented. The character of these results vary widely between the strongest and weakest sets of support conditions. In order to prevent buckling for the weakest conditions, considerable tension must be provided on parallel edges for just small amounts of compression applied on the opposite set of edges.

Additional keywords: theses, buckling equations; flat rectangular composite laminate.

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$$K_{x_{o_{tr_2}}} = -K_{y_o}\,(n+1)^2 + 1.0/(a_o/b_o)^2_{tr_2} + (n+1)^4\,(a_o/b_o)^2_{tr_2}$$

-or- (29)

$$K_{x_{o_{tr_2}}} = -K_{y_o}\,(\,n\,)^2 + 1.0/(a_o/b_o)^2_{tr_2} + (\,n\,)^4\,(a_o/b_o)^2_{tr_2}$$

where

$K_{x_{o_{tr_2}}}$ = x_o-buckling coefficient at transition from (m=1,(n+1)) to (m=1,n)

Of course, since $(a_o/b_o)_{tr_2}$ and $K_{x_{o_{tr_2}}}$ signify coordinates of a mutual point of the (m=1,n+1) and (m=1,n) curves, either equation presented in equation (29) yields the correct answer.

Consideration now is directed solely at relatively large aspect ratios, a_o/b_o. The prior work in this section showed that for comparatively small a_o/b_o, m=1 returns the minimum K_{x_o}. In contrast, setting n to its smallest integer value, one, produces the smallest K_{x_o} as a_o/b_o grows. This progression mirrors the logic presented for the case K_{y_o} less than or equal to zero. As a result, the identical equations (25) and (26) which defined the transitional quantities $(a_o/b_o)_{tr_1}$ and $K_{x_{o_{tr_1}}}$ for K_{y_o} in tension apply for the present compression conditions. No further manipulation is required.

Table II gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets based upon equations (20), (25), (26), (28), and (29). The values of m and n are clearly marked for each point, and each entry which corresponds to a transition point

is superscripted in the a_o/b_o column with a star (*). Note that K_{y_o} is greater than zero for all points since Table I presents buckling coefficient data for laminates under tension and zero K_{y_o} loading.

The statistics presented in Table II expose three important buckling characteristics of laminates compressed in the y_o-direction. First, the transition values of a_o/b_o increase as K_{y_o} grows. As the pattern for negative K_{y_o} values showed, this trend is most pronounced for the initial transition points, and its effect diminishes as a_o/b_o becomes large. Second, irrespective of the compressive magnitude of K_{y_o}, K_{x_o} attains a limiting value of 2.0 as a_o/b_o approaches infinity. Therefore, K_{x_o} is effectively independent of the magnitude of K_{y_o} -- positive or negative -- because the same asymptote of 2.0 is approached by tensile K_{y_o}. Finally, for K_{y_o} algebraically large, the K_{x_o} value drops into the tensile range for certain a_o/b_o values. Furthermore, as K_{y_o} increases, a tension load in the x_o-direction produces buckling for larger spans of the a_o/b_o dimension.

Figure 1 represents a plot of K_{x_o} versus a_o/b_o for eleven distinct values of K_{y_o}. The lowest curve characterizes $K_{y_o}=5.0$; whereas, the highest depicts $K_{y_o}=-5.0$. The nine other curves differ from each other by increments of one. This graph reinforces the concept that K_{x_o} for a constant K_{y_o} is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. Also, the tensile nature of K_{x_o} for

large K_{y_0} is illustrated by those portions of the curves which fall below $K_{x_0}=0$. Another key visual reinforcement of data which this graph provides is the merging of all curves to the $K_{x_0}=2.0$ asymptote as a_0/b_0 becomes very large.

Figure 2 plots in three dimensions the same information as Figure 1. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 1; however, the quantitative aspect of Figure 2 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Rearrangement of equation (20) produces a relation which allows visualization of selected two-dimensional slices through Figure 2 (3:356-360).

$$K_{x_0}\ m^2 + K_{y_0}\ n^2 = [m^2/(a_0/b_0)]^2 + [n^2\ (a_0/b_0)]^2 \qquad (30)$$

Thus, for constant a_0/b_0, each unique set of (m,n) yields a straight-line variation between K_{y_0} and K_{x_0}. The locus of the lowest sections of each line comprises the complete two-dimensional graph.

Table III gives selected a_0/b_0, K_{y_0}, and K_{x_0} ordered triplets based upon equation (30). Small, intermediate, and relatively large values of a_0/b_0 -- 0.8, 1.6, and 2.6, respectively -- constitute the three a_0/b_0 quantities of interest. Also included is the corresponding pair (m,n) for each point. Each entry which constitutes a point of

intersection is subscripted in the K_{y_o} column with a star (*).

Figures 3,4,and 5 represent plots of K_{x_o} versus K_{y_o} for the constant a_o/b_o values of 0.8, 1.6, and 2.6, respectively. The graphs constitute the minimum ordinates of intersecting straight line segments. Note especially that K_{x_o} declines as K_{y_o} increases and that the rate of decline of K_{x_o} for an increase in K_{y_o} jumps markedly for small a_o/b_o.

TABLE I

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported on All Sides (for K_{y_o} tensile or zero)

m	n	a_o/b_o	K_{y_o}	K_{x_o}
1	1	0.6000	-5.0	8.1378
1	1	0.7320	-5.0	7.4021
1	1	0.8376*	-5.0	7.1270
2	1	1.4834	-5.0	3.6180
2	1	2.0000*	-5.0	3.2500
3	1	2.4994	-5.0	2.6903
3	1	3.1237*	-5.0	2.5621
10	1	10.369 *	-5.0	2.0552
11	1	10.819	-5.0	2.0424
11	1	11.381 *	-5.0	2.0460
1	1	0.6000	-2.0	5.1378
1	1	0.8843	-2.0	4.0608
1	1	1.1118*	-2.0	4.0451
2	1	1.6917	-2.0	2.6806
2	1	2.2545*	-2.0	2.5577
3	1	2.7293	-2.0	2.2581
3	1	3.3229*	-2.0	2.2642
10	1	10.441 *	-2.0	2.0274
11	1	10.887	-2.0	2.0170
11	1	11.446 *	-2.0	2.0228
1	1	0.6000	0.0	3.1378
1	1	1.0523	0.0	2.0104
1	1	1.4142*	0.0	2.5000
2	1	1.9894	0.0	2.0001
2	1	2.4495*	0.0	2.1667
3	1	3.0132	0.0	2.0001
3	1	3.4641*	0.0	2.0833
10	1	10.488 *	0.0	2.0091
11	1	11.044	0.0	2.0001
11	1	11.489 *	0.0	2.0076

TABLE II

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported on All Sides (for K_{y_o} compressive)

m	n	a_o/b_o	K_{y_o}	K_{x_o}
1	3	0.2774*	1.0	10.231
1	2	0.3528	1.0	6.0241
1	2	0.4472*	1.0	4.2000
1	1	0.9598	1.0	1.0067
1	1	1.6005*	1.0	1.9519
2	1	2.0241	1.0	1.7506
2	1	2.5536*	1.0	1.9936
3	1	2.9907	1.0	1.8889
3	1	3.5370*	1.0	1.9983
10	1	10.512 *	1.0	2.0000
11	1	10.956	1.0	1.9918
11	1	11.511 *	1.0	2.0000
1	3	0.4804*	3.0	-3.9744
1	2	0.6111	3.0	-3.3466
1	2	0.7746*	3.0	-0.7333
1	1	1.4554	3.0	-0.4098
1	1	2.0000*	3.0	1.2500
2	1	2.3432	3.0	1.3511
2	1	2.7721*	3.0	1.6917
3	1	3.1787	3.0	1.6801
3	1	3.6869*	3.0	1.8391
10	1	10.560 *	3.0	1.9819
11	1	11.002	3.0	1.9752
11	1	11.555 *	3.0	1.9849
1	3	0.6202*	5.0	-11.246
1	2	0.8312	5.0	-7.4986
1	2	1.0000*	5.0	-3.0000
1	1	1.6168	5.0	-2.0034
1	1	2.3878*	5.0	0.8770
2	1	2.6599	5.0	1.0841
2	1	3.0000*	5.0	1.4444
3	1	3.3740	5.0	1.4999
3	1	3.8416*	5.0	1.6940
10	1	10.608 *	5.0	1.9639
11	1	11.158	5.0	1.9595
11	1	11.598 *	5.0	1.9699

TABLE III

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported on All Sides

m	n	a_o/b_o	K_{y_o}	K_{x_o}
1	1	0.8000	-5.0000	7.2025
1	1	0.8000	-1.5000	3.7025
1	1	0.8000	3.2000*	-0.9975
1	2	0.8000	3.5000	-2.1975
1	2	0.8000	5.0000	-8.1975
2	1	1.6000	-5.0000	3.4525
2	1	1.6000	-1.5000	2.5775
2	1	1.6000	0.9975*	1.9531
1	1	1.6000	3.5000	-0.5494
1	1	1.6000	5.0000	-2.0494
3	1	2.6000	-5.0000	2.6380
3	1	2.6000	-1.5000	2.2491
3	1	2.6000	1.4346*	1.9231
2	1	2.6000	3.5000	1.4067
2	1	2.6000	5.0000	1.0317

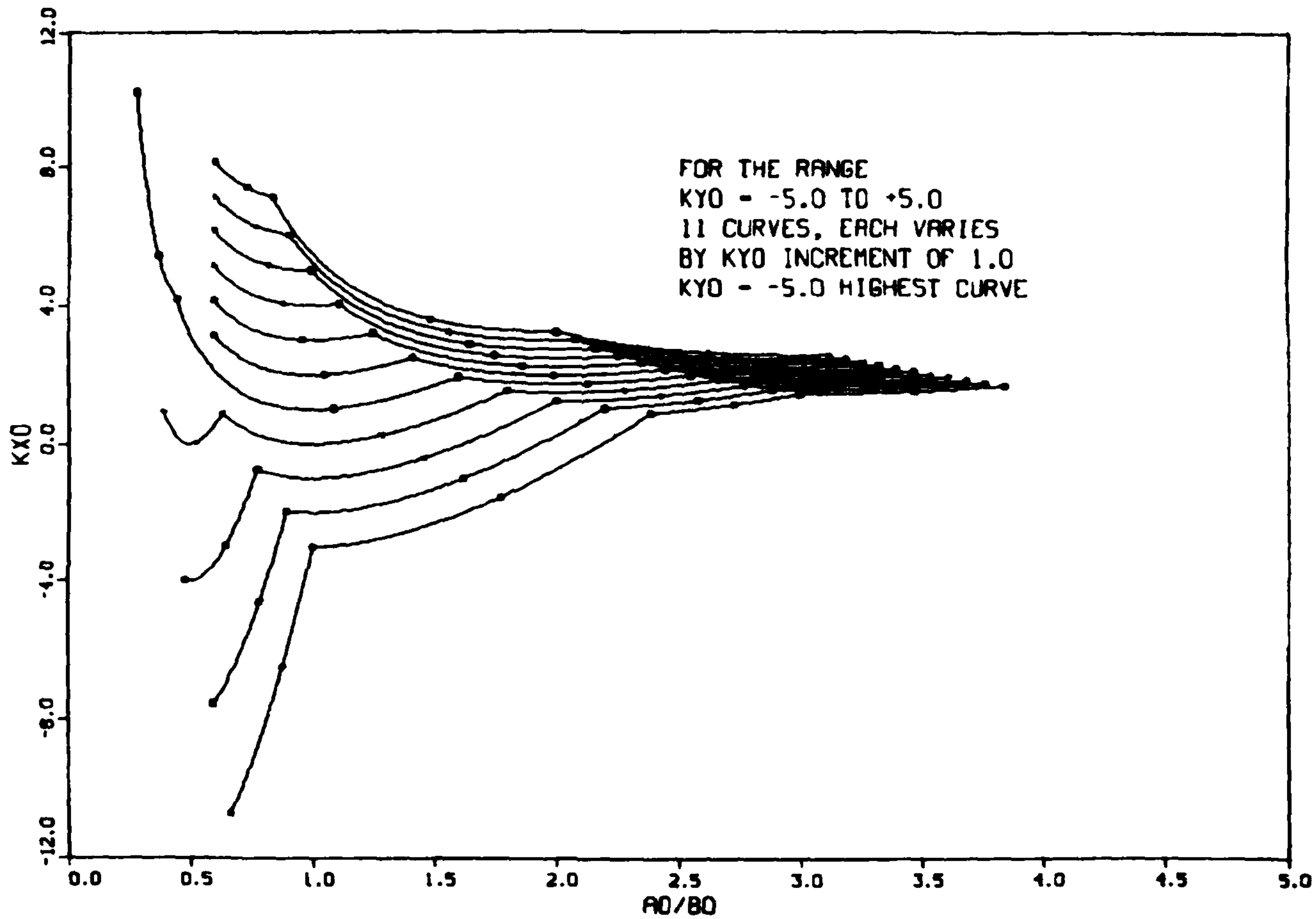

FIGURE 1

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-S-S-S LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

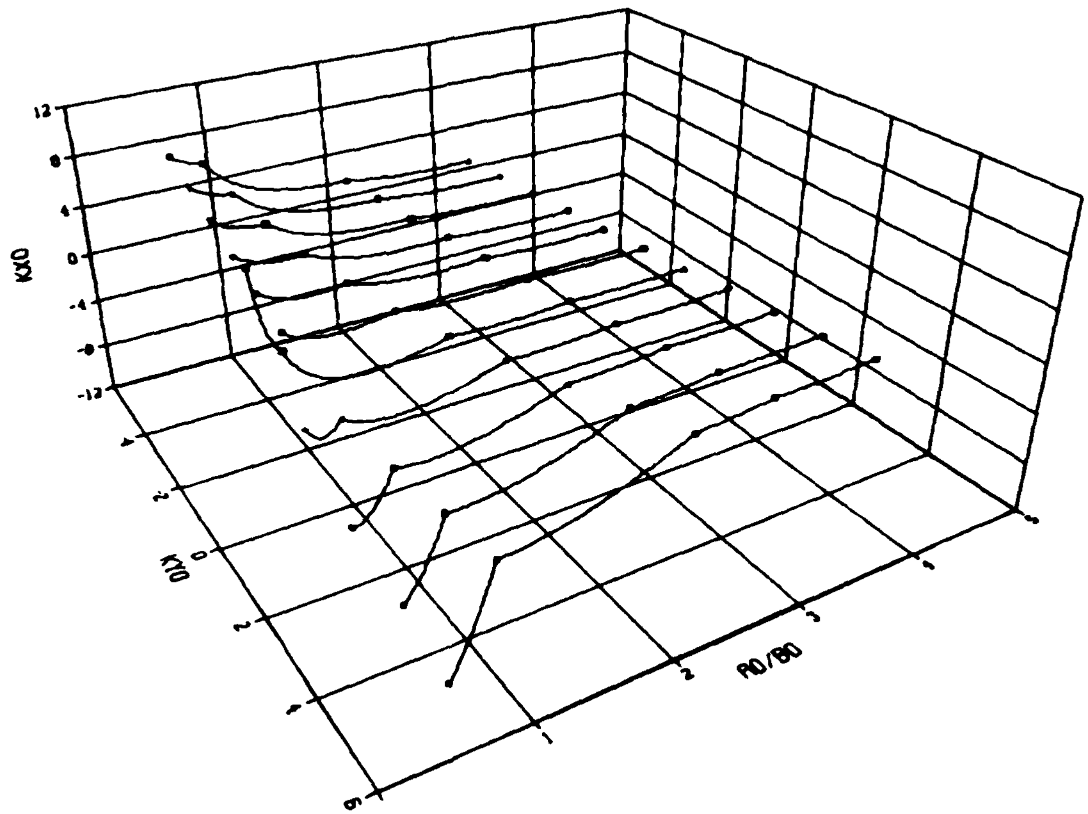

FIGURE 2

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-S-S-S LAMINATE

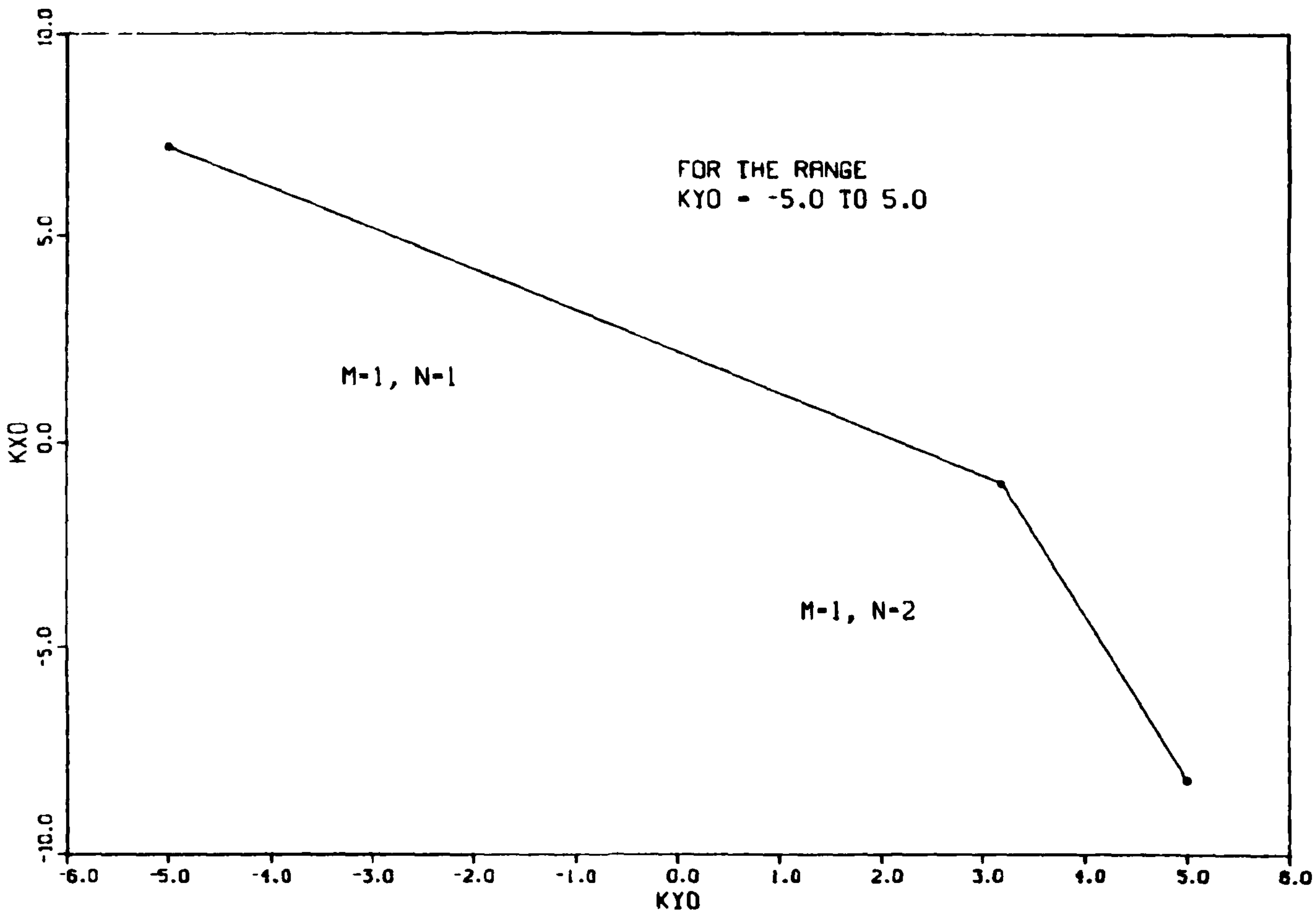

FIGURE 3

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 0.8 FOR AN S-S-S-S LAMINATE

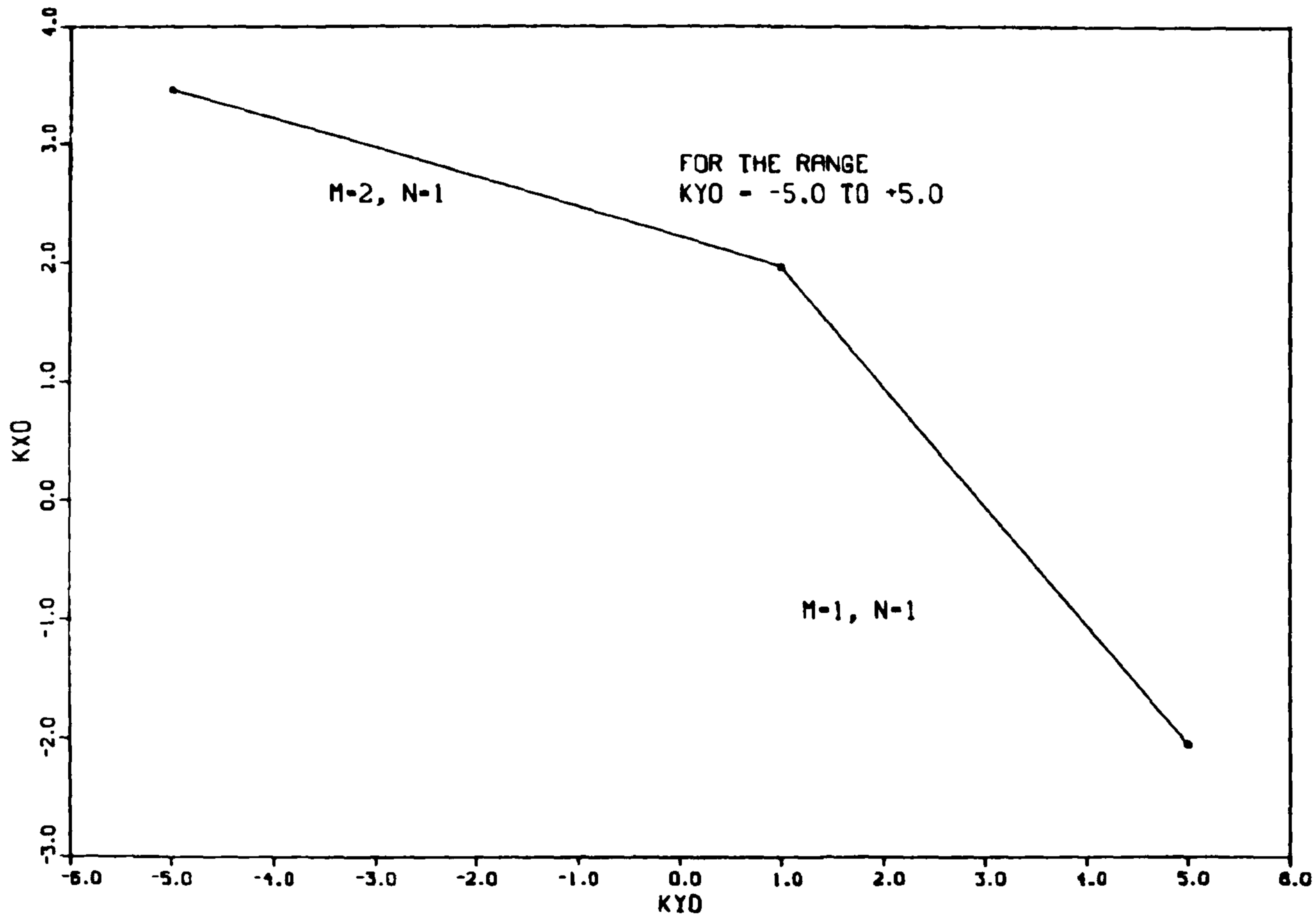

FIGURE 4

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.6 FOR AN S-S-S-S LAMINATE

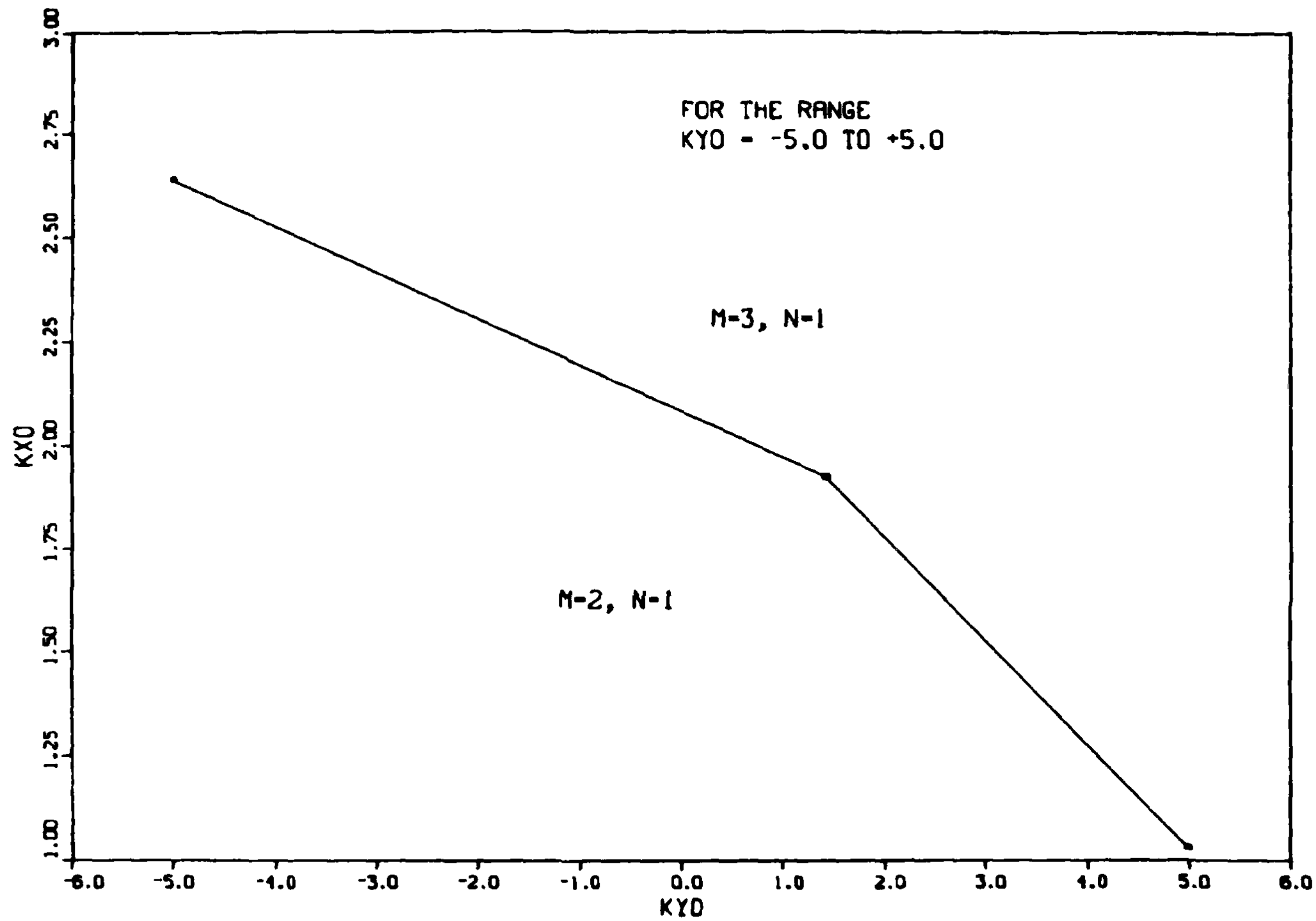

FIGURE 5

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.6 FOR AN S-S-S-S LAMINATE

III. Flat Rectangular Composite Laminate Clamped on All Sides

The boundary conditions for a laminate clamped on all sides specify that the vertical displacement and the slope of the vertical displacement with respect to the normal to each edge must vanish along each edge in the affine space. In equation form, the following must hold:

$$\begin{aligned}
&\text{on edge } x_o=-a_o/2\ , \quad w=0 \ ; \quad w_{,x_o}=0\\
&\text{on edge } x_o=\ a_o/2\ , \quad w=0 \ ; \quad w_{,x_o}=0\\
&\text{on edge } y_o=-b_o/2\ , \quad w=0 \ ; \quad w_{,y_o}=0\\
&\text{on edge } y_o=\ b_o/2\ , \quad w=0 \ ; \quad w_{,y_o}=0
\end{aligned} \tag{31}$$

Note that the origin of coordinates in the affine space is taken to be at the center of the laminate. This choice of origin location allows maximum simplicity in manipulations with the doubly symmetric boundary conditions.

For this case the general buckling equation (14) is most efficiently solved by the separation of variables technique. Therefore, the displacement w is defined by a function X, which depends only upon x_o, multiplied by a different function Y, which depends only upon y_o.

$$w = X(x_o)\ Y(y_o) \tag{32}$$

Substitution of this assumed form of w into equation (14) yields:

$$X^{IV}(x_o)\ Y(y_o) + X(x_o)\ Y^{IV}(y_o) + K_{x_o}(\pi/b_o)^2 X''(x_o)\ Y(y_o) + K_{y_o}(\pi/a_o)^2 X(x_o)\ Y''(y_o) = 0 \tag{33}$$

where

$X^{IV}(x_o)$ = the fourth standard derivative of X with respect to x_o.

$X''(x_o)$ = the second standard derivative of X with respect to x_o.

$Y^{IV}(y_o)$ = the fourth standard derivative of Y with respect to y_o.

$Y''(y_o)$ = the second standard derivative of Y with respect to y_o.

Division of equation (33) by the quantity $X(x_o)\ Y(y_o)$ gives:

$$[X^{IV}(x_o)/X(x_o) + K_{x_o}(\pi/b_o)^2 X''(x_o)/X(x_o)] + [Y^{IV}(y_o)/Y(y_o) + K_{y_o}(\pi/a_o)^2 Y''(y_o)/Y(y_o)] = 0 \qquad (34)$$

The terms enclosed in the first set of square brackets are functions solely of x_o; whereas, the components bracketed by the second set depend only on y_o. For these two groups to sum to zero, each individual group can be equal to no more than a constant. This concept is expressed by the following two equations:

$$Y^{IV}(y_o)/Y(y_o) + K_{y_o}(\pi/a_o)^2 Y''(y_o)/Y(y_o) = k^4 \qquad (35)$$

$$X^{IV}(x_o)/X(x_o) + K_{x_o}(\pi/b_o)^2 X''(x_o)/X(x_o) = -k^4 \qquad (36)$$

where

k^4 = constant. This quantity is raised to the fourth power so that fractional exponents may be avoided in the work to follow.

The constant k is now expressed in terms of another constant, f_n:

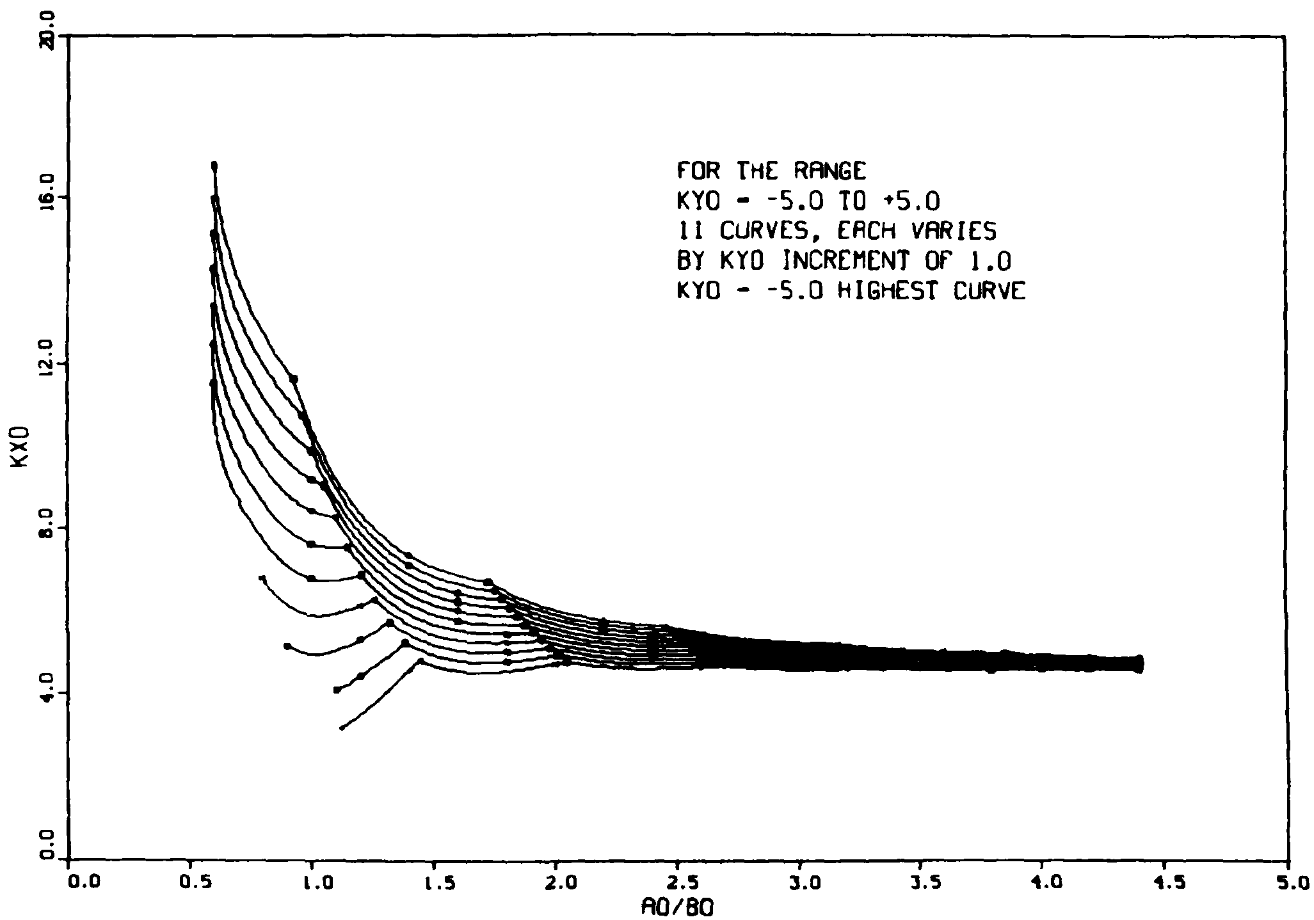

FIGURE 6

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR A C-C-C-C LAMINATE
FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

TABLE V

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Clamped on All Sides

(the first column denotes the symmetic or antisymmetric nature of f_n , and the second the symmetric or antisymmetric nature of K_{x_o}.)

f_n	x	a_o/b_o	K_{y_o}	K_{x_o}
S	A	1.2000	-5.0	8.3805
S	A	1.2000	-2.0	7.6648
S	A	1.2000	0.0	7.1740
S	S	1.2000	2.0	6.1458
S	S	1.2000	5.0	3.5231
S	S	2.4000	-5.0	5.6513
S	S	2.4000	-2.0	5.3535
S	S	2.4000	0.0	5.1476
S	S	2.4000	2.0	4.9361
S	S	2.4000	5.0	4.6087
S	S	3.6000	-5.0	5.0193
S	S	3.6000	-2.0	4.8941
S	S	3.6000	0.0	4.8094
S	S	3.6000	2.0	4.7238
S	S	3.6000	5.0	4.5933

TABLE IV

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Clamped on All Sides

(The first column denotes the symmetric or antisymmetric nature of f_n , and the second the symmetric or antisymmetric nature of K_{x_o} .)

f_n	x	a_o/b_o	K_{y_o}	K_{x_o}
S	S	0.6000	-5.0	16.780
S	S	0.9268*	-5.0	11.641
S	A	1.4000	-5.0	7.3587
S	A	1.7260*	-5.0	6.7135
S	S	2.0000	-5.0	6.0208
S	S	2.4578*	-5.0	5.6283
S	A	2.8000	-5.0	5.3151
S	A	3.1622*	-5.0	5.2001
S	S	3.6000	-5.0	5.0193
S	S	3.8531*	-5.0	4.9844
S	S	25.000	-5.0	4.5952
S	S	0.6000	-2.0	14.273
S	S	1.0511*	-2.0	9.0508
S	A	1.4000	-2.0	6.6631
S	A	1.8154*	-2.0	6.0686
S	S	2.2000	-2.0	5.4487
S	S	2.5257*	-2.0	5.3297
S	A	2.8000	-2.0	5.1190
S	A	3.2168*	-2.0	5.0253
S	S	3.6000	-2.0	4.8941
S	S	3.8986*	-2.0	4.8687
S	S	25.000	-2.0	4.5925
S	S	0.8000	2.0	6.8023
S	S	1.2611*	2.0	6.2879
S	A	1.6000	2.0	5.3003
S	A	1.9444*	2.0	5.2899
S	S	2.2000	2.0	5.0006
S	S	2.6200*	2.0	4.9530
S	A	3.0000	2.0	4.7983
S	A	3.2913*	2.0	4.8002
S	S	3.6000	2.0	4.7230
S	S	3.9603*	2.0	4.7181
S	S	25.000	2.0	4.5889
S	S	1.2000	5.0	3.5231
S	S	1.4410*	5.0	4.8158
S	A	1.8000	5.0	4.5527
S	A	2.0476*	5.0	4.7702
S	S	2.4000	5.0	4.6087
S	S	2.6933*	5.0	4.6871
S	A	3.0000	5.0	4.6034
S	A	3.3485*	5.0	4.6376
S	S	4.0074*	5.0	4.6079
S	S	25.000	5.0	4.5863

Figures 8, 9, and 10 represent two-dimensional plots at constant a_o/b_o slices of 1.2, 2.4, and 3.6, respectively. These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that K_{x_o} declines as K_{y_o} increases and that the rate of decline of K_{x_o} for an increase in K_{y_o} jumps markedly for small a_o/b_o.

eleven distinct values of K_{y_o} . The lowest curve characterizes $K_{y_o} = 5.0$; whereas, the highest depicts $K_{y_o} = -5.0$ The nine other curves differ from each other by increments of one. This graph reinforces the concept that K_{x_o} for a constant K_{y_o} is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of $K_{x_o} = 4.59$ for a_o/b_o large is readily apparent.

Figure 7 plots in three dimensions the same information as Figure 6. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 6; however, the quantitative aspect of Figure 7 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table V gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o--1.2, 2.4, and 3.6 Also included are the states of symmetry or antisymmetry for K_{x_o} and the f_n which produces this minimum K_{x_o}.

$(a_o/b_o, K_{y_o}, K_{x_o})$ based upon symmetric $X_S(x_o)$.

Comparison is now made between the triplets $(a_o/b_o, K_{y_o}, K_{x_o})$ based upon antisymmetric and symmetric $X(x_o)$. The final answer, or coordinate point, is simply the one for which the K_{x_o} is lowest. Therefore, only after this final trial is the correct K_{x_o} fixed for any a_o/b_o and K_{y_o} combination.

Table IV gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets. Also included are the states of symmetry or antisymmetry for K_{x_o} and the f_n which produces this minimum K_{x_o}. Furthermore, each entry which corresponds to a transition point from symmetric to antisymmetric buckling, or vice verse, is superscripted in the a_o/b_o column with a star (*).

The statistics presented in Table IV expose three important characteristics of laminates under compression or tension in the y_o-direction. First, the transition values of a_o/b_o increase as K_{y_o} becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as a_o/b_o becomes large. Second, irrespective of the magnitude of K_{y_o}, K_{x_o} attains a limiting value of 4.59 as a_o/b_o approaches infinity. Finally, the f_n which produces the minimum K_{x_o} is in all cases determined by the lowest value of the symmetric $Y_S(y_o)$ separation equation. This phenomenon holds for symmetric and antisymmetric K_{x_o} values.

Figure 6 represents a plot of K_{x_o} versus a_o/b_o for

of f_n determined through the antisymmetric or symmetric $Y(y_o)$ equations. The f_n of course of primary interest is the one which returns the smallest value of K_{x_o}. Thus, for any fixed set of a_o/b_o and K_{y_o}, all possible values of f_n must be considered so that the lowest K_{x_o} is found to complete the triplet $(a_o/b_o, K_{y_o}, K_{x_o})$ based upon antisymmetric $X_A(x_o)$.

In an identical manner, equations (68) and (69) are employed as boundary conditions for the symmetric portion of $X(x_o)$, $X_S(x_o)$. For non-trivial constants F_n and H_n, the following determinental equation must hold:

$$\begin{vmatrix} \cos(\omega a_o/2) & \cos(\Phi a_o/2) \\ -\omega \sin(\omega a_o/2) & -\Phi \sin(\Phi a_o/2) \end{vmatrix} = 0 \qquad (72)$$

Expansion of the determinant and multiplication by the quantity $(-a_o/2)$ gives:

$$(\Phi a_o/2) \sin(\Phi a_o/2) \cos(\omega a_o/2) - (\omega a_o/2) \sin(\omega a_o/2) \cos(\Phi a_o/2) = 0 \qquad (73)$$

Solution of equation (73) for the infinite number of sets of values of $\omega a_o/2$ and $\Phi a_o/2$ determines another infinite set of values of K_{x_o} for any choice of f_n, a_o/b_o, and K_{y_o}. Again, the f_n may be any member from those groups of f_n returned by the antisymmetric or symmetric $Y(y_o)$ equations. Primary interest of course centers upon the value of f_n which yields the smallest K_{x_o}. So as before, for any fixed set of a_o/b_o and K_{y_o}, all possible values of f_n must be considered so that the minimum K_{x_o} is found to complete the triplet

$$X(-a_o/2)\,Y(y_o) = 0 \quad ; \quad X'(-a_o/2)\,Y(y_o) = 0$$

$$X(a_o/2)\,Y(y_o) = 0 \quad ; \quad X'(a_o/2)\,Y(y_o) = 0 \tag{67}$$

The two lines of equation (67) each express identical information when the $X(x_o)$ function is broken down into its components $X_A(x_o)$ and $X_S(x_o)$. Thus, only the bottom line of information of equation (67) will be manipulated. For equation (67) to hold for non-trivial $Y(y_o)$, the following boundary conditions on the function $X(x_o)$ must hold:

$$X(a_o/2) = 0 \tag{68}$$

$$X'(a_o/2) = 0 \tag{69}$$

Application of equations (68) and (69) first is made to the antisymmetric portion of $X(x_o)$, $X_A(x_o)$. For non-trivial constants E_n and G_n, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\omega a_o/2) & \sin(\Phi a_o/2) \\ \omega\cos(\omega a_o/2) & \Phi\cos(\Phi a_o/2) \end{vmatrix} = 0 \tag{70}$$

Expansion of the determinant and multiplication by the quantity $a_o/2$ gives:

$$(\Phi a_o/2)\sin(\omega a_o/2)\cos(\Phi a_o/2) - (\omega a_o/2)\sin(\Phi a_o/2)\cos(\omega a_o/2) = 0 \tag{71}$$

Solution of equation (71) for the infinite number of sets of values of $\omega a_o/2$ and $\Phi a_o/2$ equivalently yields an infinite set of values of K_{x_o} for any choice of f_n, a_o/b_o, and K_{y_o}. Note that the f_n may be any member from the groups

In a similar fashion, choice of the minus sign just before the first square bracket in equation (60) yields the final two values of g. For this choice of the negative sign, the entire quantity in the curly brackets is again constrained to be greater than zero. So the final two values of g are:

$$g_{3,4} = \pm\, i(\pi/b_o)\ \{\ K_{x_o}/2 - [(K_{x_o}/2)^2 - 16\ f_n^4\ ]^{1/2}\ \}^{1/2} \qquad (62)$$

By the theory of linear homogeneous equations, the function $X(x_o)$ can be easily determined.

$$X(x_o) = N_n e^{g_1 x_o} + P_n e^{g_2 x_o} + Q_n e^{g_3 x_o} + R_n e^{g_4 x_o} \qquad (63)$$

where

N_n, P_n, Q_n, R_n = arbitrary constants

Equivalently, equation (63) can be expressed as:

$$X(x_o) = E_n \sin(\omega x_o) + F_n \cos(\omega x_o) + G_n \sin(\Phi x_o) + H_n \cos(\Phi x_o) \qquad (64)$$

where

$\omega = (\pi/b_o)\ \{\ K_{x_o}/2 + [(K_{x_o}/2)^2 - 16\ f_n^4\ ]^{1/2}\ \}^{1/2}$

$\Phi = (\pi/b_o)\ \{\ K_{x_o}/2 - [(K_{x_o}/2)^2 - 16\ f_n^4\ ]^{1/2}\ \}^{1/2}$

E_n, F_n, G_n, H_n = another set of arbitrary constants

The function $X(x_o)$ likewise can be further simplified by reduction into its antisymmetric and symmetric parts.

$$X_A(x_o) = E_n \sin(\omega x_o) + G_n \sin(\Phi x_o) \qquad (65)$$

$$X_S(x_o) = F_n \cos(\omega x_o) + H_n \cos(\Phi x_o) \qquad (66)$$

Consider the boundary conditions, equations (31), for this case of a laminate clamped on all sides. The initial two, expressed in the separation functions, become:

equation (37) into equation (36) and slight rearrangement yields:

$$X^{IV}(x_o) + K_{x_o}(\pi/b_o)^2 X''(x_o) + 16\ f_n^4 (\pi/b_o)^4 X(x_o) = 0 \qquad (56)$$

Similar to equation (38), equation (55) is a linear, homogeneous differential equation with constant coefficients. Assume the following form of $X(x_o)$:

$$X(x_o) = e^{gx_o} \qquad (57)$$

where

g = constant

Substitution of equation (57) into equation (56) gives:

$$\{\ g^4 + K_{x_o}(\pi/b_o)^2 g^2 + 16\ f_n^4 (\pi/b_o)^4\ \}\ e^{gx_o} = 0 \qquad (58)$$

For equation (58) to hold, the terms inside the brackets must sum to zero. This fact allows determination of the four values of g which satisfy equation (58). First, g^2 can be determined by the quadratic equation.

$$g^2 = 0.5\ \{\ -K_{x_o}(\pi/b_o)^2 \pm [\ K_{x_o}^2 (\pi/b_o)^4 - 64\ f_n^4 (\pi/b_o)^4\ ]^{1/2}\ \} \qquad (59)$$

$$g^2 = (\pi/b_o)^2\ i^2\ \{\ K_{x_o}/2 \pm [\ (K_{x_o}/2)^2 - 16\ f_n^4\ ]^{1/2}\ \} \qquad (60)$$

The first two values of g can be determined by choice of the positive, as opposed to the negative, sign just before the first square bracket in equation (60). For this choice of the positive sign, the entire quantity in the curly brackets is constrained to be greater than zero. Therefore, the initial two of the four desired values of g are:

$$g_{1,2} = \pm\ i(\pi/b_o)\ \{\ K_{x_o}/2 + [(K_{x_o}/2)^2 - 16\ f_n^4\ ]^{1/2}\ \}^{1/2} \qquad (61)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$(\rho b_o/2)\ \sin(\eta b_o/2)\ \cosh(\rho b_o/2)$$
$$- (\eta b_o/2)\ \cos(\eta b_o/2)\ \sinh(\rho b_o/2) = 0 \qquad (53)$$

Solution of equation (53) for the infinite number of sets of values $\eta b_o/2$ and $\rho b_o/2$ equivalently yields an infinite set of constants f_n $n=1,2,...$ for any choice of K_{y_o} and a_o/b_o. These values of f_n will be utilized in the $X(x_o)$ functional equations.

In an identical manner, equations (50) and (51) are employed as boundary conditions for the symmetric portion of $Y(y_o)$, $Y_S(y_o)$. For non-trivial constants B_n and D_n, the following determinental equation must hold:

$$\begin{vmatrix} \cos(\eta b_o/2) & \cosh(\rho b_o/2) \\ -\eta \sin(\eta b_o/2) & \rho \sinh(\rho b_o/2) \end{vmatrix} = 0 \qquad (54)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$(\rho b_o/2)\ \cos(\eta b_o/2)\ \sinh(\rho b_o/2)$$
$$+ (\eta b_o/2)\ \sin(\eta b_o/2)\ \cosh(\rho b_o/2) = 0 \qquad (55)$$

Solution of equation (55) for the infinite number of values $\eta b_o/2$ and $\rho b_o/2$ determines another infinite set of constants f_n $n=1,2,...$ for any choice of K_{y_o} and a_o/b_o. The importance of this group will be evident shortly.

Consider now equation (36), the differential equation which will render the $X(x_o)$ function. Substitution of

A_n, B_n, C_n, D_n = another arbitrary set of constants

The function $Y(y_o)$ can be further simplified by reduction into its antisymmetric and symmetric parts.

$$Y_A(y_o) = A_n \sin(\eta y_o) + C_n \sinh(\rho y_o) \tag{47}$$

$$Y_S(y_o) = B_n \cos(\eta y_o) + D_n \cosh(\rho y_o) \tag{48}$$

Consider the boundary conditions, equations (31), for this case of a laminate clamped on all sides. The final two, expressed in the separation functions, become:

$$\begin{aligned} X(x_o)\, Y(-b_o/2) = 0 \quad &; \quad X(x_o)\, Y'(-b_o/2) = 0 \\ X(x_o)\, Y(\,b_o/2) = 0 \quad &; \quad X(x_o)\, Y'(\,b_o/2) = 0 \end{aligned} \tag{49}$$

The two lines of equation (49) each express identical information when the function $Y(y_o)$ is broken down into its components $Y_A(y_o)$ and $Y_S(y_o)$. Thus, only the bottom line of information of equation (49) will be manipulated. For equation (49) to hold for non-trivial $X(x_o)$, the following boundary conditions on the function $Y(y_o)$ must hold:

$$Y(b_o/2) = 0 \tag{50}$$

$$Y'(b_o/2) = 0 \tag{51}$$

Application of equations (50) and (51) first is made to the antisymmetric portion of $Y(y_o)$, $Y_A(y_o)$. For non-trivial constants A_n and C_n , the following determinental equation must hold:

$$\begin{vmatrix} \sin(\eta b_o/2) & \sinh(\rho b_o/2) \\ \eta \cos(\eta b_o/2) & \rho \cosh(\rho b_o/2) \end{vmatrix} = 0 \tag{52}$$

the first square bracket in equation (42). For this choice of the positive sign, the entire quantity in the curly brackets is constrained to be greater than zero. Therefore, the initial two of four desired values of r are:

$$r_{1,2} = \pm(\pi/a_o)\, i \;\; \{K_{y_o}/2 + [(K_{y_o}/2)^2 + 16\, f_n^4 (a_o/b_o)^4]^{1/2}\}^{1/2} \qquad (43)$$

In a similar fashion, choice of the minus sign just before the first square bracket in equation (42) yields the final two values of r. For this selection, however, the quantity contained in the curly brackets is negative. Multiplication of the factor i^2 outside the curly brackets with each term inside those braces rectifies the situation and allows a square root of the entire equation (42) to be taken. This procedure yields the last two values of r:

$$r_{3,4} = \pm(\pi/a_o) \{-K_{y_o}/2 + [(K_{y_o}/2)^2 + 16\, f_n^4 (a_o/b_o)^4]^{1/2}\}^{1/2} \qquad (44)$$

By the theory of linear homogeneous equations, the function $Y(y_o)$ can be easily determined.

$$Y(y_o) = J_n e^{r_1 y_o} + K_n e^{r_2 y_o} + L_n e^{r_3 y_o} + M_n e^{r_4 y_o} \qquad (45)$$

where

J_n, K_n, L_n, M_n = arbitrary constants

Equivalently, equation (45) can be expressed as:

$$Y(y_o) = A_n \sin(\eta y_o) + B_n \cos(\eta y_o) + C_n \sinh(\rho y_o) + D_n \cosh(\rho y_o) \qquad (46)$$

where

$$\eta = (\pi/a_o)\{ K_{y_o}/2 + [(K_{y_o}/2)^2 + 16\, f_n^4 (a_o/b_o)^4]^{1/2}\}^{1/2}$$

$$\rho = (\pi/a_o)\{-K_{y_o}/2 + [(K_{y_o}/2)^2 + 16\, f_n^4 (a_o/b_o)^4]^{1/2}\}^{1/2}$$

$$k = 2\, f_n (\pi/b_o) \tag{37}$$

Substitution of equation (37) into equation (35) and slight rearrangement yields:

$$Y^{IV}(y_o) + K_{y_o}(\pi/a_o)^2 Y''(y_o) - 16\, f_n^4 (\pi/b_o)^4 Y(y_o) = 0 \tag{38}$$

Equation (38) is merely a linear, homogeneous differential equation with constant coefficients. The classic solution procedure is to predict that $Y(y_o)$ fits the following:

$$Y(y_o) = e^{r y_o} \tag{39}$$

where

e = natural base = 2.71828...

r = constant

Substitution of equation (39) into equation (38) gives:

$$\{\, r^4 + K_{y_o}(\pi/a_o)^2 r^2 - 16\, f_n^4 (\pi/b_o)^4 \,\}\, e^{r y_o} = 0 \tag{40}$$

For equation (40) to hold, the terms in the brackets must sum to zero. This fact allows determination of the four values of r which satisfy equation (40). First, r^2 can be determined by the quadratic equation.

$$r^2 = 0.5\, \{-K_{y_o}(\pi/a_o)^2 \pm [K_{y_o}^2 (\pi/a_o)^4 + 64\, f_n^4 (\pi/b_o)^4\,]^{1/2}\,\} \tag{41}$$

$$r^2 = (\pi/a_o)^2 i^2\, \{K_{y_o}/2 \pm [\,(K_{y_o}/2)^2 + 16\, f_n^4 (a_o/b_o)^4\,]^{1/2}\,\} \tag{42}$$

where

$i = (-1)^{1/2}$

The first two values of r can be determined by choice of the positive, as opposed to the negative, sign just before

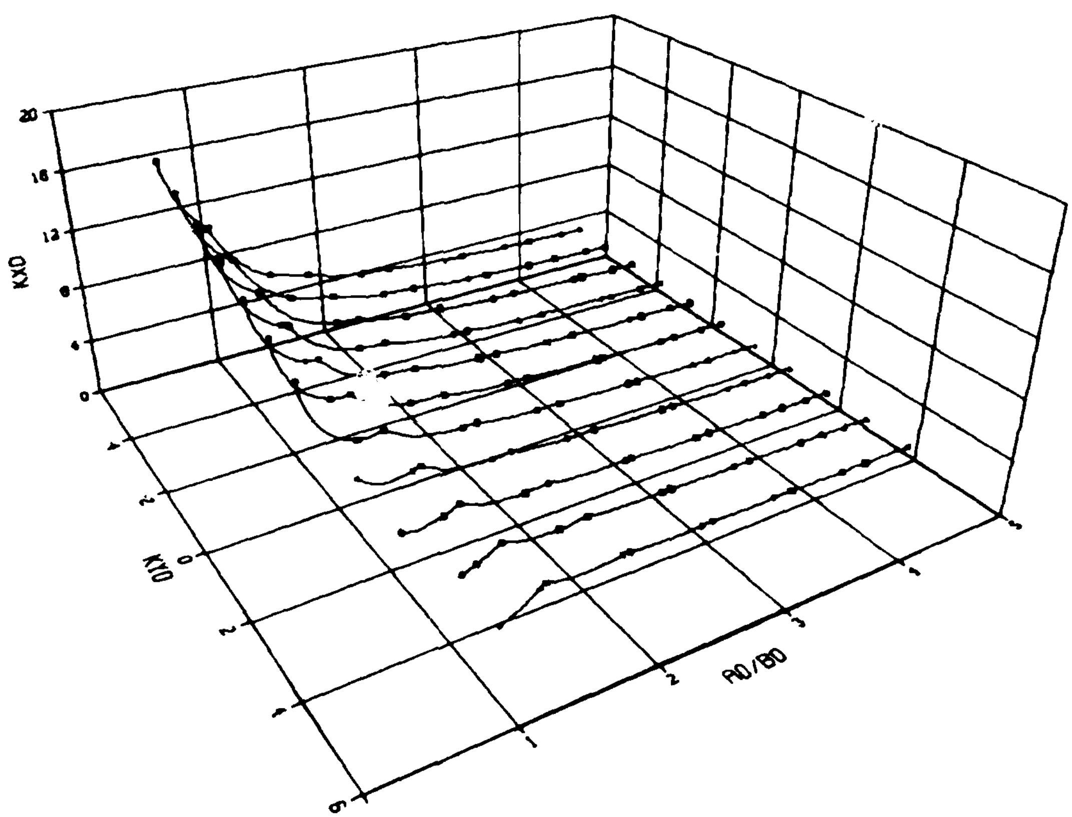

FIGURE 7

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFINE ASPECT RATIO FOR A C-C-C-C LAMINATE

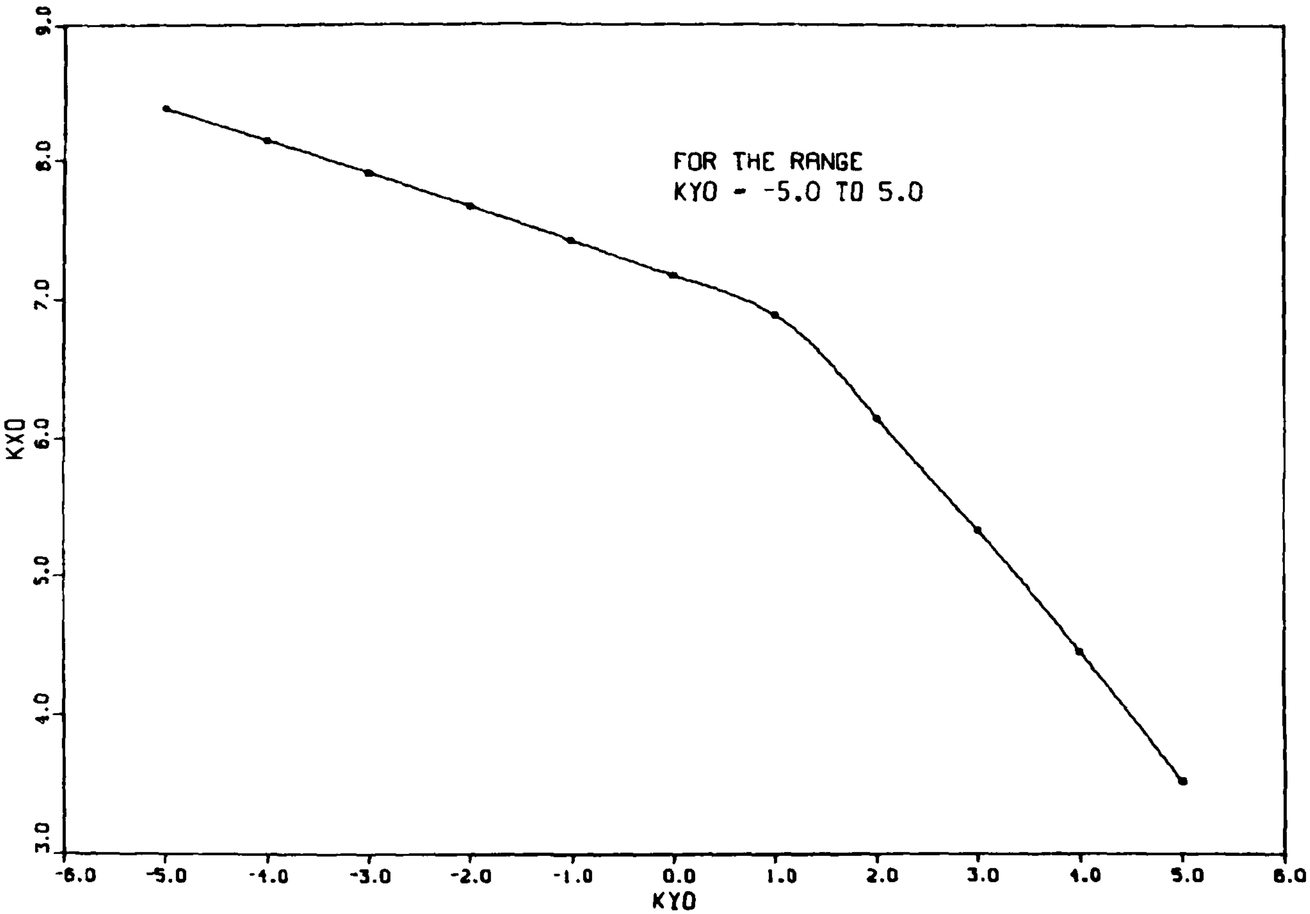

FIGURE 8

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.2 FOR A C-C-C-C LAMINATE

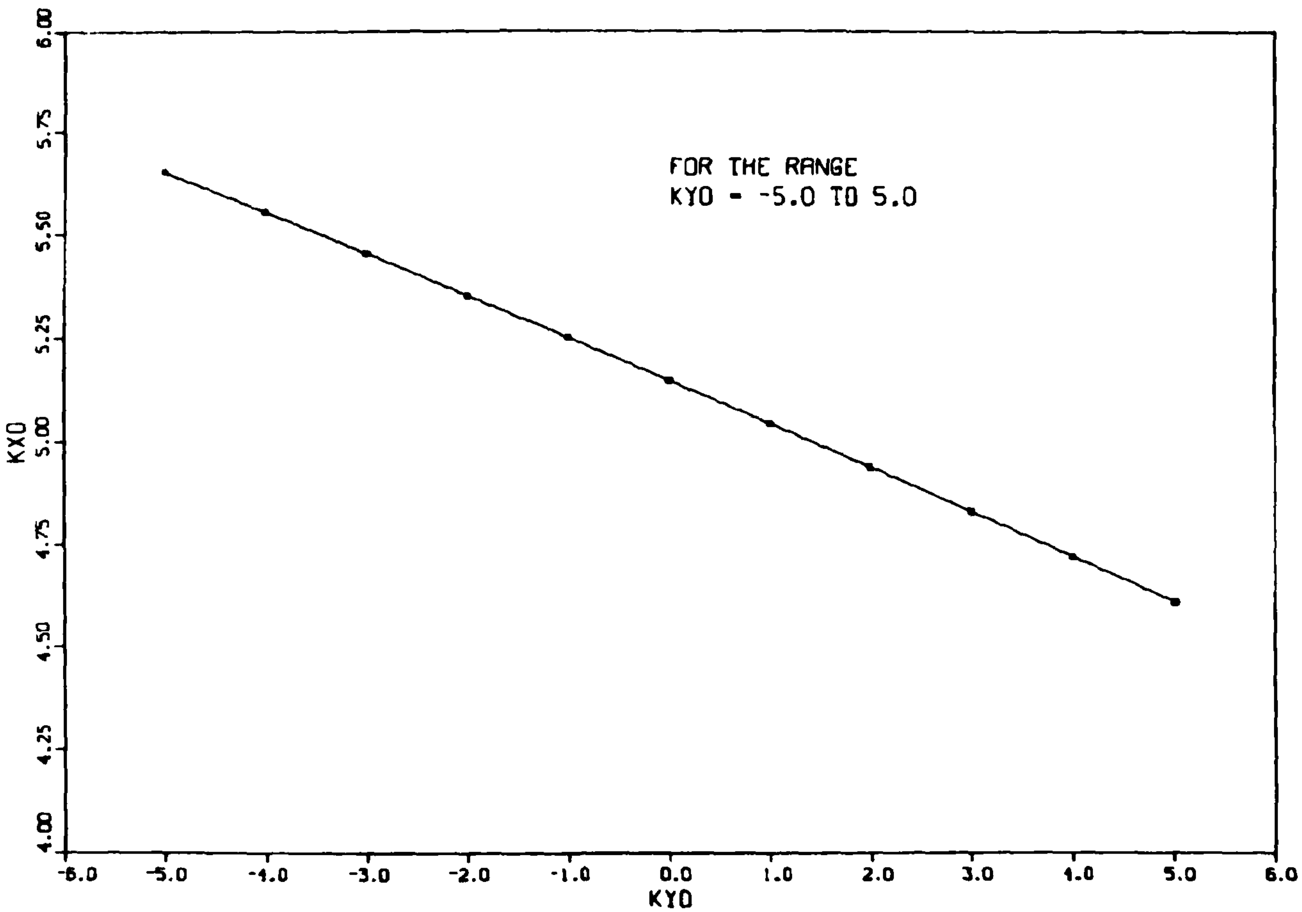

FIGURE 9

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.4 FOR A C-C-C-C LAMINATE

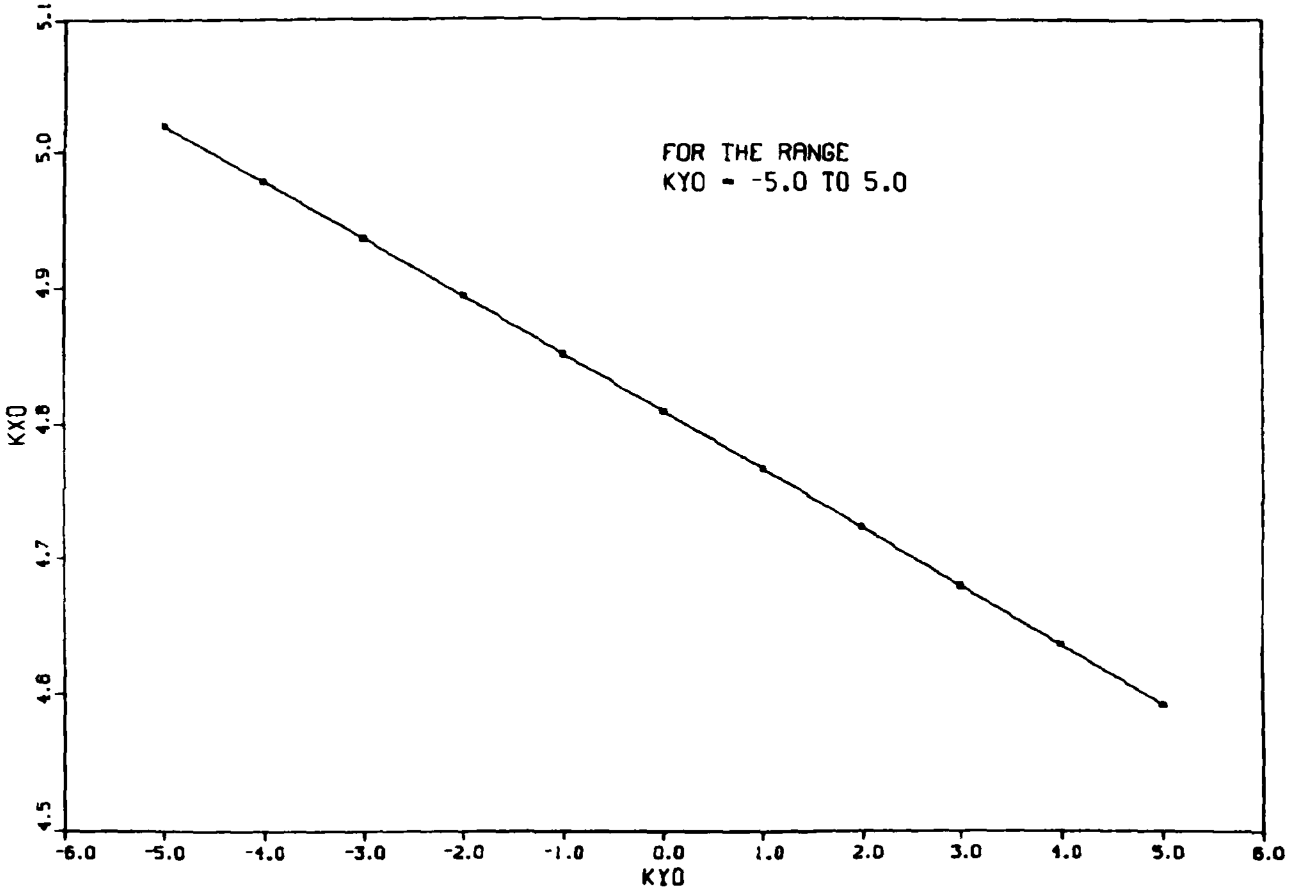

FIGURE 10

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 3.6 FOR A C-C-C-C LAMINATE

IV. Flat Rectangular Composite Laminate Simply Supported in the x_o-Direction and Clamped in the y_o-Direction

The boundary conditions for a laminate simply supported in the x_o-direction and clamped in the y_o-direction are not the same on each edge as in the previous work. For the two edges which have normals parallel to the x_o-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. On the other hand, for the two edges which have normals parallel to the y_o-axis, the vertical displacement and the slope of the vertical displacement with respect to y_o must vanish along each edge. In equation form, the following must hold:

$$
\begin{aligned}
&\text{on edge } x_o=-a_o/2\ , \quad w = 0 \quad ; \quad w_{,x_o x_o} = 0 \\
&\text{on edge } x_o= a_o/2\ , \quad w = 0 \quad ; \quad w_{,x_o x_o} = 0 \\
&\text{on edge } y_o=-b_o/2\ , \quad w = 0 \quad ; \quad w_{,y_o} = 0 \\
&\text{on edge } y_o= b_o/2\ , \quad w = 0 \quad ; \quad w_{,y_o} = 0
\end{aligned}
\qquad (74)
$$

Note that the origin of coordinates in the affine space is taken to be the center of the laminate. This choice of origin location, in general, allows maximum simplicity in manipulations with the doubly symmetric boundary conditions. For one set of derivations, however, a different origin location will be used and of course will be noted.

The general buckling equation (14) is most efficiently solved for this case by the separation of variables

technique. Therefore, the displacement w is defined by a function X, which depends only upon x_o, multiplied by a different function Y, which depends only upon y_o.

$$w = X(x_o)\ Y(y_o) \tag{75}$$

An admissible function $X(x_o)$ can be easily determined for a laminate simply supported along both edges perpendicular to the x_o-direction. The following function identically satisfies the first two conditions of equations (74):

$$X(x_o) = \sin(m\pi x_o/a_o) \tag{76}$$

Substitution of equation (76) into equation (75) yields:

$$w = Y(y_o)\ \sin(m\pi x_o/a_o) \tag{77}$$

Likewise, substitution of this assumed form of w into equation (14) gives:

$$\sin(m\pi x_o/a_o)\ \{(m\pi/a_o)^4 Y(y_o) + Y^{IV}(y_o) - (m\pi/a_o)^2 K_{x_o}(\pi/b_o)^2 Y(y_o) + K_{y_o}(\pi/a_o)^2 Y''(y_o)\} = 0 \tag{78}$$

For equation (78) to hold in general, the terms inside the curly brackets must sum to zero. Therefore, equation (78) is merely a linear homogeneous differential equation with constant coefficients. The form of $Y(y_o)$ is predicted to be that of equation (39). Insertion of this value of $Y(y_o)$ into equation (78) yields:

$$[(m\pi/a_o)^4 + r^4 - (m\pi/a_o)^4 K_{x_o}(a_o/mb_o)^2 + (K_{y_o}/m^2)\ (m\pi/a_o)^2\ r^2\ ]\ e^{ry_o} = 0 \tag{79}$$

Just as above, for the right-hand side of equation (79) to vanish, the terms in the square brackets must cancel. This fact enables determination of r^2 by use of the quadratic

equation.

$$r^2 = 0.5 \{ -(K_{y_o}/m^2)\ (m\pi/a_o)^2 \pm [(K_{y_o}/m^2)^2 (m\pi/a_o)^4 + 4\ (m\pi/a_o)^4\ (K_{x_o}(a_o/mb_o)^2 - 1)]^{1/2} \} \qquad (80)$$

$$r^2 = (m\pi/a_o)^2\ \{ -(K_{y_o}/2m^2)^2 \pm [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2} \} \qquad (81)$$

The next step in the solution process depends exclusively upon the algebraic sign of the quantity in the square brackets in equation (81). Real square roots, of course, can only be taken of non-negative numbers. As a result, a separate investigation is now made for each of the three possible signs of this governing term--negative, zero, and positive. In addition, consideration of the cases in just this sequence is vital, for the desired value of K_{x_o} for any K_{y_o} and a_o/b_o combination is the smallest possible x_o-buckling coefficient. So for m constant, a K_{x_o} root found in the region $\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$ supercedes any root found in the zero or positive regions.

$\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$

For the quantity $\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$ equation (81) becomes:

$$r^2 = (m\pi/a_o)^2\ \{ -K_{y_o}/2m^2 \pm i\ [\ 1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2\ ]^{1/2} \} \qquad (82)$$

The first two values of r stem from choice of the positive sign before the imaginary number i in equation (82), while the final two can be obtained through selection of the

negative sign.

$$r_{1,2}^2 = (m\pi/a_o)^2 \{ -K_{y_o}/2m^2 + i [1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2]^{1/2} \} \tag{83}$$

$$r_{3,4}^2 = (m\pi/a_o)^2 \{ -K_{y_o}/2m^2 - i [1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2]^{1/2} \} \tag{84}$$

Consider first equation (83). It is desired to express the right-hand side of this equation in polar form.

$$r_{1,2}^2 = M e^{ip} \tag{85}$$

where

M = the modulus of the complex number

p = angle which is the amplitude of the complex number

$$M = (m\pi/a_o)^2 \{ (-K_{y_o}/2m^2)^2 + [1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2] \}^{1/2} \tag{86}$$

$$M = (m\pi/a_o)^2 \{ 1 - K_{x_o}(a_o/mb_o)^2 \}^{1/2} \tag{87}$$

$$p = \arctan \{ [1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2]^{1/2} / (-K_{y_o}/2m^2) \} \tag{88}$$

Note especially that the modulus M is independent of K_{y_o}.

For later use, note the following:

$$-\tan(p) = \tan(-p) = \{ - [1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2]^{1/2} / (-K_{y_o}/2m^2) \} \tag{89}$$

$$-p = \arctan \{ -[1 - K_{x_o}(a_o/mb_o)^2 - (K_{y_o}/2m^2)^2]^{1/2} / (-K_{y_o}/2m^2) \} \tag{90}$$

Now consider equation (84). From equations (86) and (90) it is easy to see that the equation specifying the third and fourth roots of r can be written in polar form as:

$$r^2_{3,4} = M\ e^{-ip} \tag{91}$$

Since the quantity p is merely an angle, p has the same value as $(p + 2\pi)$. This fact allows separation of roots one and two and roots three and four.

$$r^2_1 = M\ e^{ip} \tag{92}$$

$$r^2_2 = M\ e^{i(p+2\pi)} \tag{93}$$

$$r^2_3 = M\ e^{-ip} \tag{94}$$

$$r^2_4 = M\ e^{-i(p+2\pi)} \tag{95}$$

Now, determination of the actual values of each r can be carried out.

$$r_1 = M^{1/2}\ e^{ip/2} = M^{1/2} \{ \cos(p/2) + i \sin(p/2) \} \tag{96}$$

$$r_2 = M^{1/2}\ e^{i(p+2\pi)/2}$$

$$= M^{1/2} \{ \cos(p/2 + \pi) + i \sin(p/2 + \pi) \} \tag{97}$$

$$r_2 = M^{1/2} \{ -\cos(p/2) - i \sin(p/2) \} \tag{98}$$

$$r_3 = M^{1/2} e^{-ip/2} = M^{1/2} \{ \cos(-p/2) + i \sin(-p/2) \} \tag{99}$$

$$r_3 = M^{1/2} \{ \cos(p/2) - i \sin(p/2) \} \tag{100}$$

$$r_4 = M^{1/2} e^{-i(p+2\pi)/2}$$

$$= M^{1/2} \{ \cos(-p/2 - \pi) + i \sin(-p/2 - \pi) \} \tag{101}$$

$$r_4 = M^{1/2} \{ -\cos(p/2) + i \sin(p/2) \} \tag{102}$$

Note especially that r_1 and r_3 are complex conjugate pairs, as are r_2 and r_4 .

Define the following variables:

$$c = M^{1/2} \cos(p/2) \tag{103}$$

$$s = M^{1/2} \sin(p/2) \tag{104}$$

Since the four roots of r have been fixed, the function Y(y) now reads:

$$Y(y_o) = J_m e^{r_1 y_o} + K_m e^{r_2 y_o} + L_m e^{r_3 y_o} + M_m e^{r_4 y_o} \tag{105}$$

where

J_m, K_m, L_m, M_m = arbitrary constants

Equivalently, equation (105), when combined with equations (103) and (104) can be expressed as:

$$Y(y_o) = e^{cy_o} \{ A_m \sin(sy_o) + B_m \cos(sy_o) \}$$
$$+ e^{-cy_o} \{ C_m \sin(sy_o) + D_m \cos(sy_o) \} \tag{106}$$

where

A_m, B_m, C_m, D_m = another set of arbitrary constants

Consider now the boundary conditions, equations (74), for this case of a laminate simply supported in the x_o-direction and clamped in the y_o-direction. As indicated previously, the origin location is subject to change, and the present situation dictates such a movement for maximum ease in manipulations. In particular, the x_o-origin remains the same, but the y_o-origin drops from the center of the plate to one edge. The final two equations of (74) then become:

$$Y(0) \sin(m\pi x_o/a_o) = 0 \quad ; \quad Y'(0) \sin(m\pi x_o/a_o) = 0 \tag{107}$$

$$Y(b_o) \sin(m\pi x_o/a_o) = 0 \quad ; \quad Y'(b_o) \sin(m\pi x_o/a_o) = 0 \tag{108}$$

For equations (107) and (108) to have meaning in the general case, the following conditions must hold:

$$Y(0) = 0 \tag{109}$$

$$Y'(0) = 0 \tag{110}$$

$$Y(b_o) = 0 \tag{111}$$

$$Y'(b_o) = 0 \tag{112}$$

First, apply equation (109) to equation (106). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (110) on the $Y(y_o)$ equation similarly determines a value C_m in terms of A_m and B_m.

$$sA_m + cB_m + sC_m - cD_m = 0 \tag{113}$$

But since $D_m = - B_m$

$$C_m = -A_m - 2B_m\ (c/s) \tag{114}$$

For these values of C_m and D_m, equation (106) takes on the following form:

$$\begin{aligned} Y(y_o) &= A_m\{ \sin(sy_o)\ (e^{cy_o} - e^{-cy_o})\} \\ &+ B_m\{-2(c/s)\ e^{-cy_o} \sin(sy_o) + \cos(sy_o)\ (e^{cy_o} - e^{-cy_o})\} \end{aligned} \tag{115}$$

The first derivative of $Y(y_o)$ with respect to y_o is then easy to obtain.

$$\begin{aligned} Y'(y_o) = A_m\{ &\sin(sy_o)\ (ce^{cy_o} + ce^{-cy_o}) \\ + &\cos(sy_o)\ (se^{cy_o} - se^{-cy_o})\} \\ + B_m\{ &\sin(sy_o)\ (se^{-cy_o} - se^{cy_o} + 2(c^2/s)e^{-cy_o}) \\ + &\cos(sy_o)\ (ce^{cy_o} - ce^{-cy_o})\} \end{aligned} \tag{116}$$

Substitution of equations (115) and (116) into equations (111) and (112) yield two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(sb_o)(e^{cb_o} - e^{-cb_o}) & \sin(sb_o)(-(2c/s)e^{-cb_o}) \\ & + \cos(sb_o)(e^{cb_o} - e^{-cb_o}) \\ \hline \sin(sb\ )(ce^{cb_o} + ce^{-cb_o}) & \sin(sb_o)(-se^{cb_o} + se^{-cb_o} \\ & +2(c\ /s)e^{-cb_o}) \\ + \cos(sb_o)(se^{cb_o} - se^{-cb_o}) & + \cos(sb_o)(ce^{cb_o} - ce^{-cb_o}) \end{vmatrix} = 0 \qquad (117)$$

Expansion of the determinant gives:

$$e^{2cb_o} - 2 + e^{-2cb_o} - 4(c/s)^2 \sin^2(sb_o) = 0 \qquad (118)$$

Slight rearrangement and multiplication of the final term on the left-hand side of the equation by $(b_o/b_o)^2$ yields a transcendental equation in the variables a_o/b_o, K_{y_o}, and K_{x_o}.

$$e^{2cb_o} + e^{-2cb_o} - 2\{ 1 + 2(cb_o/sb_o)^2 \sin^2(sb_o) \} = 0 \qquad (119)$$

Unfortunately, for any combination of a_o/b_o and K_{y_o}, no value of K_{x_o} satisfies equation (119). In other words, no possible solutions exist for the present set of boundary conditions for $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ equation (81) becomes:

$$r^2 = (m\pi/a_o)^2 (-K_{y_o}/2m^2) = (\pi/a_o)^2 (-K_{y_o}/2) \qquad (120)$$

Since the character of equation (120) differs drastically for the choice of algebraic sign of K_{y_o}, each possible range of K_{y_o} --negative, zero, and positive--will be analyzed as separate subcases.

$K_{y_o} < 0$.

Two values of r which satisfy equation (120) for $K_{y_o} < 0$ are straightforward.

$$r_{1,2} = \pm (\pi/a_o) (-K_{y_o}/2)^{1/2} \tag{121}$$

For simplicity, make the following definition:

$$T = (\pi/a_o) (-K_{y_o}/2)^{1/2} \tag{122}$$

From equations (121) and (122), the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function $Y(y_o)$ can be determined.

$$Y(y_o) = A_m \sinh(Ty_o) + B_m \cosh(Ty_o) + C_m y_o \sinh(Ty_o) + D_m y_o \cosh(Ty_o) \tag{123}$$

where

A_m, B_m, C_m, D_m = set of arbitrary constants

Equations (109) through (112) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (109) on equation (123) dictates that B_m must vanish. In addition, application of equation (110) means that $D_m = -A_m T$ So equation (123) takes the following form:

$$Y(y_o) = A_m\{\sinh(Ty_o) - Ty_o \cosh(Ty_o)\} + C_m y_o \sinh(Ty_o) \tag{124}$$

The first derivative of $Y(y_o)$ with respect to y_o is therefore easy to obtain.

$$Y'(y_o) = -A_m T^2 y_o \sinh(Ty_o) + C_m\{\sinh(Ty_o) + Ty_o \cosh(Ty_o)\} \tag{125}$$

Substitution of equations (124) and (125) into equations (111) and (112) again yield two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the

following determinental equation must hold:

$$\begin{vmatrix} \sinh(Tb_o) - Tb_o\cosh(Tb_o) & b_o\sinh(Tb_o) \\ -T^2 b_o\sinh(Tb_o) & \sinh(Tb_o) + Tb_o\cosh(Tb_o) \end{vmatrix} = 0 \qquad (126)$$

Expansion of the determinant gives:

$$\sinh^2(Tb_o) \quad - \quad (Tb_o)^2 \quad = 0 \qquad (127)$$

No value of Tb_o greater than zero can satisfy equation (127). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} < 0$

$K_{y_o} = 0$.

One value of r which satisfies equation (120) for $K_{y_o} = 0$ is simply $r_1 = 0$ From this simple result, the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function $Y(y_o)$ can be determined.

$$Y(y_o) = A_m + B_m y_o + C_m y_o^2 + D_m y_o^3 \qquad (128)$$

where

A_m, B_m, C_m, D_m = set of arbitrary constants

Equations (109) through (112) constitute the set of constraints for the $Y(y_o)$ function. Equations (109) and (110) imply that $A_m = B_m = 0$ Equation (111) and equation (112), with each side of (112) multiplied by b_o, yield the following set of simultaneous equations.

$$\begin{aligned} C_m b_o^2 + D_m b_o^3 &= 0 \\ 2C_m b_o^2 + 3D_m b_o^3 &= 0 \end{aligned} \qquad (129)$$

Only $C_m = D_m = 0$ constitutes a valid solution for

TABLE VI

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Clamped on the Two Edges Normal to the y_o-Direction

(The second column denotes the symmetric or antisymmetric nature of K_{x_o}.)

m	x	a_o/b_o	K_{y_o}	K_{x_o}
1	S	0.6000	-3.0	8.2337
1	S	0.7720*	-3.0	8.3892
2	S	1.0000	-3.0	6.2049
2	S	1.2000	-3.0	5.5520
2	S	1.5199*	-3.0	5.6273
3	S	1.8000	-3.0	5.0410
3	S	2.0000	-3.0	4.9476
3	S	2.2234*	-3.0	5.0572
4	S	2.4000	-3.0	4.8607
4	S	2.6000	-3.0	4.7711
4	S	2.9098*	-3.0	4.8423
1	S	0.6000	0.0	4.6277
1	S	0.8000	0.0	4.8513
1	S	0.9393*	0.0	5.6672
2	S	1.2000	0.0	4.6277
2	S	1.4000	0.0	4.5588
2	S	1.6269*	0.0	4.9116
3	S	1.8000	0.0	4.6277
3	S	2.0000	0.0	4.5339
3	S	2.3008*	0.0	4.7227
4	S	2.6000	0.0	4.5380
4	S	2.8000	0.0	4.5588
4	S	2.9703	0.0	4.6471
1	S	1.0000	3.0	2.3188
1	S	1.1515*	3.0	3.7707
2	S	1.4000	3.0	3.6145
2	S	1.6000	3.0	3.9094
2	S	1.7430*	3.0	4.2789
3	S	2.0000	3.0	4.1164
3	S	2.2000	3.0	4.2059
3	S	2.3814*	3.0	4.4082
4	S	2.6000	3.0	4.3036
4	S	2.8000	3.0	4.3245
4	S	3.0323*	3.0	4.4590

however, the quantitative aspect of Figure 12 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table VII gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o -- 1.2, 2.4, and 3.6. Also included are the states of symmetry or antisymmetry for K_{x_o}.

Figure 13, 14, and 15 represent two-dimensional plots at constant a_o/b_o slices of 1.2, 2.4, and 3.6, respectively. These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that K_{x_o} declines as K_{y_o} increases and that the rate of decline of K_{x_o} for an increase in K_{y_o} jumps markedly for small a_o/b_o.

the state of symmetry for K_{x_o} and the integer value of m which produces this minimum K_{x_o}. Furthermore, each entry point which correponds to a transition point from the m curve to the (m+1) curve is superscripted in the a_o/b_o column with a star (*).

The statistics presented in Table VI expose two important characteristics of laminates under compression or tension in the y_o-direction. First, the transition values of a_o/b_o increase as K_{y_o} becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as a_o/b_o approaches a large number. Second, irrespective of the magnitude of K_{y_o}, K_{x_o} attains a limiting value of 4.534 as a_o/b_o approaches infinity.

Figure 11 represents a plot of K_{x_o} versus a_o/b_o for eleven distinct values of K_{y_o}. The lowest curve characterizes $K_{y_o} = 5.0$; whereas, the highest depicts $K_{y_o} = -5.0$ The nine other curves differ from each other by increments of one. This graph reinforces the concept that K_{x_o} for a constant K_{y_o} is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of $K_{x_o} = 4.534$ for a_o/b_o large is readily apparent.

Figure 12 plots in three dimensions the same information as Figure 11. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 11;

$$(\beta_m b_o/2)\sinh(\theta_m b_o/2)\cosh(\beta_m b_o/2)$$
$$-(\theta_m b_o/2)\sinh(\beta_m b_o/2)\cosh(\theta_m b_o/2) = 0 \qquad (180)$$

Attempts at solution of this antisymmetric buckling equation (180) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o} . Therefore, further consideration of this equation is dropped.

In an identical fashion, equations (150) and (151) are employed as boundary conditions for the symmetric portion of $Y(y_o)$, $Y_S(y_o)$. For non-trivial constants B_m and D_m , the following determinental equation must hold:

$$\begin{vmatrix} \cosh(\theta_m b_o/2) & \cosh(\beta_m b_o/2) \\ \theta_m \sinh(\theta_m b_o/2) & \beta_m \sinh(\beta_m b_o/2) \end{vmatrix} = 0 \qquad (181)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$(\beta_m b_o/2)\cosh(\theta_m b_o/2)\sinh(\beta_m b_o/2)$$
$$-(\theta_m b_o/2)\sinh(\theta_m b_o/2)\cosh(\beta_m b_o/2) = 0 \qquad (182)$$

Attempts at solution of this antisymmetric buckling equation (182) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o} . As a result, further considerations of this equation and this subcase as a whole are abandoned.

Discussion of Results

Table VI gives selected a_o/b_o , K_{y_o} , and K_{x_o} ordered triplets as determined by equation (169). Also included is

J_m, K_m, L_m, M_m = set of arbitrary constants

Equivalently, equation (174) can be written as:

$$Y(y_o) = A_m \sinh(\theta_m y_o) + B_m \cosh(\theta_m y_o) + C_m \sinh(\beta_m y_o) + D_m \cosh(\beta_m y_o) \quad (175)$$

where

$$\theta_m = (m\pi/a_o)\{-K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \quad (176)$$

β_m = (defined in equation (163))

A_m, B_m, C_m, D_m = another set of arbitrary constants which depend upon the integer m

The function $Y(y_o)$ can be further simplified by reduction into its antisymmetric and symmetric parts.

$$Y_A(y_o) = A_m \sinh(\theta_m y_o) + C_m \sinh(\beta_m y_o) \quad (177)$$

$$Y_S(y_o) = B_m \cosh(\theta_m y_o) + D_m \cosh(\beta_m y_o) \quad (178)$$

As illustrated in the first of these subcases, the boundary conditions shown in equations (150) and (151) govern. Application of equations (150) and (151) first is made to the antisymmetric portion of $Y(y_o)$, $Y_A(y_o)$. For non-trivial constants A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(\theta_m b_o/2) & \sinh(\beta_m b_o/2) \\ \theta_m \cosh(\theta_m b_o/2) & \beta_m \cosh(\beta_m b_o/2) \end{vmatrix} = 0 \quad (179)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

Attempts at solution of equation (169) for any combination of a_o/b_o and K_{y_o} do yield the smallest values of K_{x_o}. The buckling is always symmetric in nature. Further consideration of results generated by equation (169) is postponed until the last of the three subcases is presented.

K_{y_o} <u>Ranges from a Relatively Large Negative Number to Negative Infinity</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, K_{y_o} can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form then, the search for a solution for K_{x_o} for this subcase is limited by the following inequalities:

$$(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \quad 0 \qquad (170)$$

$$-K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2} \quad 0 \qquad (171)$$

As a result, all four values of r can be determined.

$$r_{1,2} = \pm (m\pi/a_o) \{-K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \qquad (172)$$

$$r_{3,4} = \pm (m\pi/a_o) \{-K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \qquad (173)$$

Since all four values of r are known, the desired function $Y(y_o)$ can be written as:

$$Y(y_o) = J_m e^{r_1 y_o} + K_m e^{r_2 y_o} + L_m e^{r_3 y_o} + M_m e^{r_4 y_o} \qquad (174)$$

where

antisymmetric portion of $Y(y_o)$, $Y_A(y_o)$. For non-trivial constants A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o/2) & \sinh(\beta_m b_o/2) \\ \alpha_m \cos(\alpha_m b_o/2) & \beta_m \cosh(\beta_m b_o/2) \end{vmatrix} = 0 \qquad (166)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$(\beta_m b_o/2)\ \sin(\alpha_m b_o/2)\ \cosh(\beta_m b_o/2) - (\alpha_m b_o/2)\ \cos(\alpha_m b_o/2)\ \sinh(\beta_m b_o/2) = 0 \qquad (167)$$

Attempts at solution of this antisymmetric buckling equation (167) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. Therefore, further consideration of this equation is dropped.

In an identical manner, equations (150) and (151) are employed as boundary conditions for the symmetric portion of $Y(y_o)$, $Y_S(y_o)$. For non-trivial constants B_m and D_m, the following determinental equation must hold:

$$\begin{vmatrix} \cos(\alpha_m b_o/2) & \cosh(\beta_m b_o/2) \\ -\alpha_m \sin(\alpha_m b_o/2) & \beta_m \sinh(\beta_m b_o/2) \end{vmatrix} = 0 \qquad (168)$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$(\beta_m b_o/2)\ \cos(\alpha_m b_o/2)\ \sinh(\beta_m b_o/2) + (\alpha_m b_o/2)\ \sin(\alpha_m b_o/2)\ \cosh(\beta_m b_o/2) = 0 \qquad (169)$$

As a result, all four values of r can be determined.

$$r_{1,2} = \pm (m\pi/a_o)\ i\ \{K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \quad (159)$$

$$r_{3,4} = \pm (m\pi/a_o)\ \{-K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \quad (160)$$

Since all four values of r are known, the desired function $Y(y_o)$ can be written as:

$$Y(y_o) = J_m e^{r_1 y_o} + K_m e^{r_2 y_o} + L_m e^{r_3 y_o} + M_m e^{r_4 y_o} \quad (161)$$

where

J_m, K_m, L_m, M_m = set of arbitrary constants

Equivalently, equation (161) can be expressed as:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + B_m \cos(\alpha_m y_o) + C_m \sinh(\beta_m y_o) + D_m \cosh(\beta_m y_o) \quad (162)$$

where

α_m = (defined in equation (145))

$$\beta_m = (m\pi/a_o)\{ -K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \quad (163)$$

A_m, B_m, C_m, D_m = another set of arbitrary constants which depend upon the integer m

The function $Y(y_o)$ can be further simplified by reduction into its antisymmetric and symmetric parts.

$$Y_A(y_o) = A_m \sin(\alpha_m y_o) + C_m \sinh(\beta_m y_o) \quad (164)$$

$$Y_S(y_o) = B_m \cos(\alpha_m y_o) + D_m \cosh(\beta_m y_o) \quad (165)$$

As illustrated in the previous subcase, the boundary conditions shown in equations (150) and (151) govern. Application of equations (150) and (151) first is made to the

$$\begin{vmatrix} \cos(\alpha_m b_o/2) & \cos(\gamma_m b_o/2) \\ -\alpha_m \sin(\alpha_m b_o/2) & -\gamma_m \sin(\gamma_m b_o/2) \end{vmatrix} = 0 \qquad (154)$$

Expansion of the determinant and multiplication by the quantity $(-b_o/2)$ gives:

$$(\gamma_m b_o/2)\ \sin(\gamma_m b_o/2)\ \cos(\alpha_m b_o/2)$$
$$-\ (\alpha_m b_o/2)\ \sin(\alpha_m b_o/2)\ \cos(\gamma_m b_o/2) = 0 \qquad (155)$$

Likewise, attempts at solution of equation (155) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase as a whole are abandoned.

K_{y_o} Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, K_{y_o} can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that K_{y_o} range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution fo- K_{x_o} for this subcase is limited by the following three inequalities:

$$(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 > 0 \qquad (156)$$

$$-K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2} < 0 \qquad (157)$$

$$-K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2} > 0 \qquad (158)$$

information of equation (149) will be manipulated. For equation (149) to hold in general, the following boundary conditions must be obeyed:

$$Y(b_o/2) = 0 \tag{150}$$

$$Y'(b_o/2) = 0 \tag{151}$$

Application of equations (150) and (151) first is made to the antisymmetric portion of $Y(y_o)$, $Y_A(y_o)$. For non-trivial constants A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o/2) & \sin(\gamma_m b_o/2) \\ \alpha_m \cos(\alpha_m b_o/2) & \gamma_m \cos(\gamma_m b_o/2) \end{vmatrix} = 0 \tag{152}$$

Expansion of the determinant and multiplication by the quantity $b_o/2$ gives:

$$\begin{aligned} &(\gamma_m b_o/2) \sin(\alpha_m b_o/2) \cos(\gamma_m b_o/2) \\ &- (\alpha_m b_o/2) \sin(\gamma_m b_o/2) \cos(\alpha_m b_o/2) = 0 \end{aligned} \tag{153}$$

Attempts at solution of equation (153) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. Therefore, further consideration of this equation is dropped.

In an identical manner, equations (150) and (151) are employed as boundary conditions for the symmetric portion of $Y(y_o)$, $Y_S(y_o)$. For non-trivial constants B_m and D_m, the following determinental equation must hold:

where

J_m, K_m, L_m, M_m = set of arbitrary constants

Equivalently, equation (143) can be expressed as:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + B_m \cos(\alpha_m y_o) + C_m \sin(\gamma_m y_o) + D_m \cos(\gamma_m y_o) \qquad (144)$$

where

$$\alpha_m = (m\pi/a_o)\{K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \qquad (145)$$

$$\gamma_m = (m\pi/a_o)\{K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o) - 1]^{1/2}\}^{1/2} \qquad (146)$$

A_m, B_m, C_m, D_m = another set of arbitrary constants which depend on the integer m

The function $Y(y_o)$ can be further simplified by reduction into its antisymmetric and symmetric parts.

$$Y_A(y_o) = A_m \sin(\alpha_m y_o) + C_m \sin(\gamma_m y_o) \qquad (147)$$

$$Y_S(y_o) = B_m \cos(\alpha_m y_o) + D_m \cos(\gamma_m y_o) \qquad (148)$$

Consider the boundary conditions, equations (74), for this case of a laminate simply supported in the x_o-direction and clamped in the y_o-direction. The final two, expressed in the separation functions, become:

$$Y(-b_o/2)\sin(m\pi x_o/a_o) = 0 \quad ; \quad Y'(-b_o/2)\sin(m\pi x_o/a_o) = 0$$
$$(149)$$
$$Y(b_o/2)\sin(m\pi x_o/a_o) = 0 \quad ; \quad Y'(b_o/2)\sin(m\pi x_o/a_o) = 0$$

The two lines of equation (149) each express identical information when the function $Y(y_o)$ is broken down into its components $Y_A(y_o)$ and $Y_S(y_o)$. Thus, only the bottom line of

imply not only different solution forms but different domains of K_{y_o} for valid solutions. Three subcases again must be considered so that a solution for K_{x_o} may be determined for any range of K_{y_o}.

K_{y_o} <u>Ranges from a Relatively Large Positive Number to Positive Infinity</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, K_{y_o} can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely this last quoted domain. In equation form the search for a solution for K_{x_o} for this subcase is limited by the following two inequalities:

$$(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 > 0 \qquad (139)$$

$$-K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2} < 0 \qquad (140)$$

As a result, all four values of r can be determined.

$$r_{1,2} = \pm (m\pi/a_o)\, i\, \{K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \qquad (141)$$

$$r_{3,4} = \pm (m\pi/a_o)\, i\, \{K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\}^{1/2} \qquad (142)$$

Since all four values of r are known, the desired function $Y(y_o)$ can be written as:

$$Y(y_o) = J_m e^{r_1 y_o} + K_m e^{r_2 y_o} + L_m e^{r_3 y_o} + M_m e^{r_4 y_o} \qquad (143)$$

coefficients A_m and C_m . For non-trivial A_m and C_m , the following determinental equation must hold:

$$\begin{vmatrix} \sin(Ub_o) - Ub_o\cos(Ub_o) & b_o\sin(Ub_o) \\ U^2 b_o \sin(Ub_o) & \sin(Ub_o) + Ub_o\cos(Ub_o) \end{vmatrix} = 0 \qquad (135)$$

Expansion of the determinant gives:

$$\sin^2(Ub_o) \quad - (Ub_o)^2 \quad = 0 \qquad (136)$$

No value of Ub_o greater than zero can satisfy equation (136). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} > 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$ equation (81) does not need to be altered. The first two roots of r stem from the selection of the negative sign before the left square bracket in equation (81), while the final two can be obtained by choice of the positive sign.

$$r^2_{1,2} = (m\pi/a_o)^2 \{-K_{y_o}/2m^2 - [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\} \qquad (137)$$

$$r^2_{3,4} = (m\pi/a_o)^2 \{-K_{y_o}/2m^2 + [(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1]^{1/2}\} \qquad (138)$$

The quantities of intense interest now are those contained in the curly brackets of equations (137) and (138). Positive or negative characters of each of these quantities

equations (129). So again no possible solutions exist for the present boundary conditions for $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} = 0$

$K_{y_o} > 0$.

Two values of r which satisfy equation (120) for $K_{y_o} > 0$ are straightforward.

$$r_{1,2} = \pm i\ (\pi/a_o)\ (K_{y_o}/2)^{1/2} \tag{130}$$

For simplicity, make the following definition:

$$U = (\pi/a_o)\ (K_{y_o}/2)^{1/2} \tag{131}$$

From equations (130) and (131), the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function $Y(y_o)$ can be determined.

$$Y(y_o) = A_m \sin(Uy_o) + B_m \cos(Uy_o) + C_m y_o \sin(Uy_o) + D_m y_o \cos(Uy_o) \tag{132}$$

where

A_m, B_m, C_m, D_m = set of arbitrary constants

Equations (109) through (112) again comprise the body of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (109) on equation (132) dictates that B_m must vanish. Furthermore, application of equation (110) means that $D_m = -A_m U$ So equation (132) takes the following form:

$$Y(y_o) = A_m\{\sin(Uy_o) - Uy_o \cos(Uy_o)\} + C_m y_o \sin(Uy_o) \tag{133}$$

The first derivative of $Y(y_o)$ with respect to y_o is therefore easy to obtain.

$$Y'(y_o) = A_m U^2 y_o \sin(Uy_o) + C_m\{\sin(Uy_o) + Uy_o \cos(Uy_o)\} \tag{134}$$

Substitution of equations (133) and (134) into equations (111) and (112) yield two homogeneous linear equations in

TABLE VII

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported in the x_o-Direction and Clamped in the y_o-Direction

(The second column denotes the symmetric or antisymmetric nature of K_{x_o}.)

m	x	a_o/b_o	K_{y_o}	K_{x_o}
2	S	1.2000	-5.0	6.1580
2	S	1.2000	-3.0	5.5520
2	S	1.2000	-1.0	4.9381
2	S	1.2000	0.0	4.6277
2	S	1.2000	1.0	4.3148
2	S	1.2000	3.0	3.6796
1	S	1.2000	5.0	1.7018
4	S	2.4000	-5.0	5.0153
4	S	2.4000	-3.0	4.8607
4	S	2.4000	-1.0	4.7056
4	S	2.4000	0.0	4.6277
4	S	2.4000	1.0	4.5497
4	S	2.4000	3.0	4.3933
3	S	2.4000	5.0	4.1550
6	S	3.6000	-5.0	4.8005
6	S	3.6000	-3.0	4.7315
5	S	3.6000	-1.0	4.6428
5	S	3.6000	0.0	4.5930
5	S	3.6000	1.0	4.5431
5	S	3.6000	3.0	4.4432
5	S	3.6000	5.0	4.3431

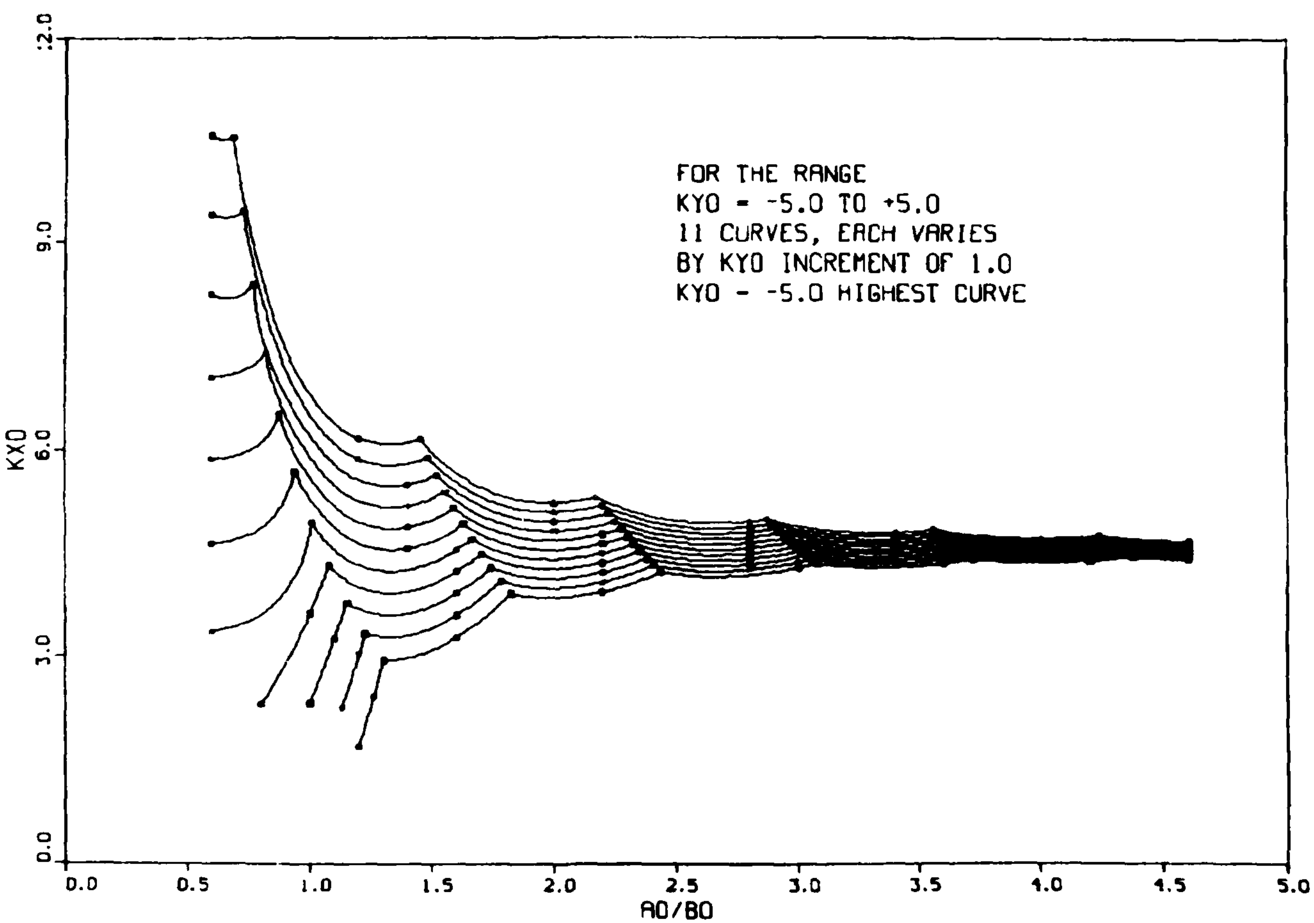

FIGURE 11

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-C-S-C LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

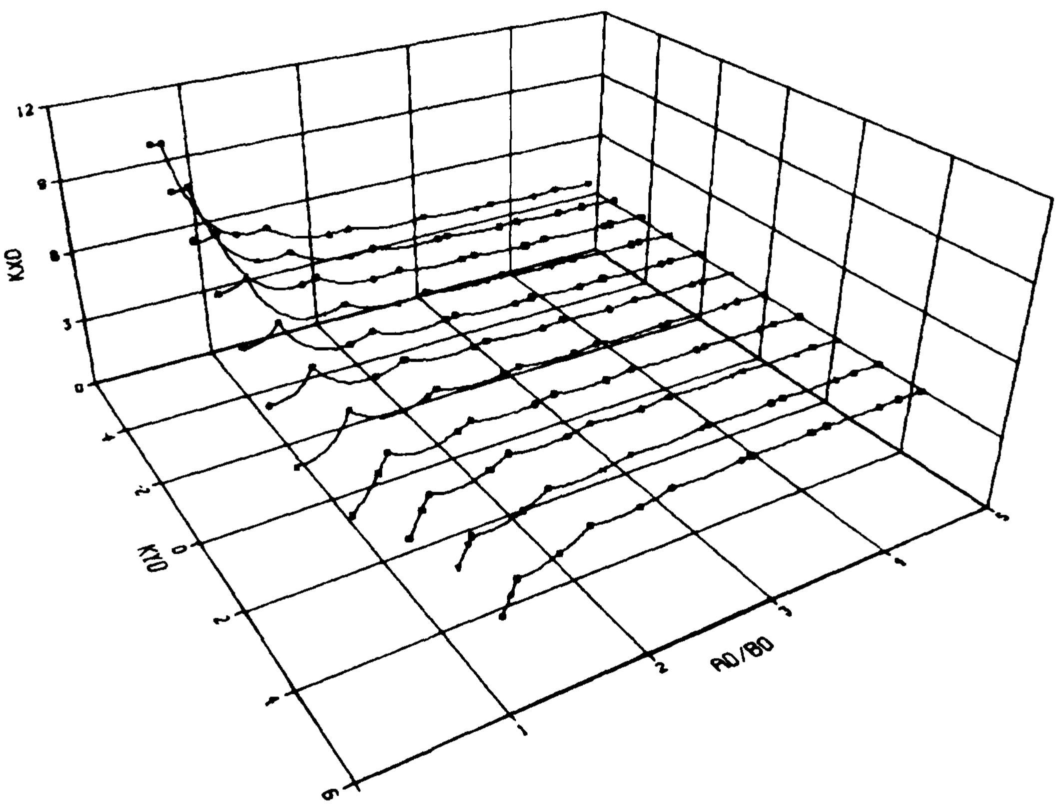

FIGURE 12

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO FOR AN S-C-S-C LAMINATE

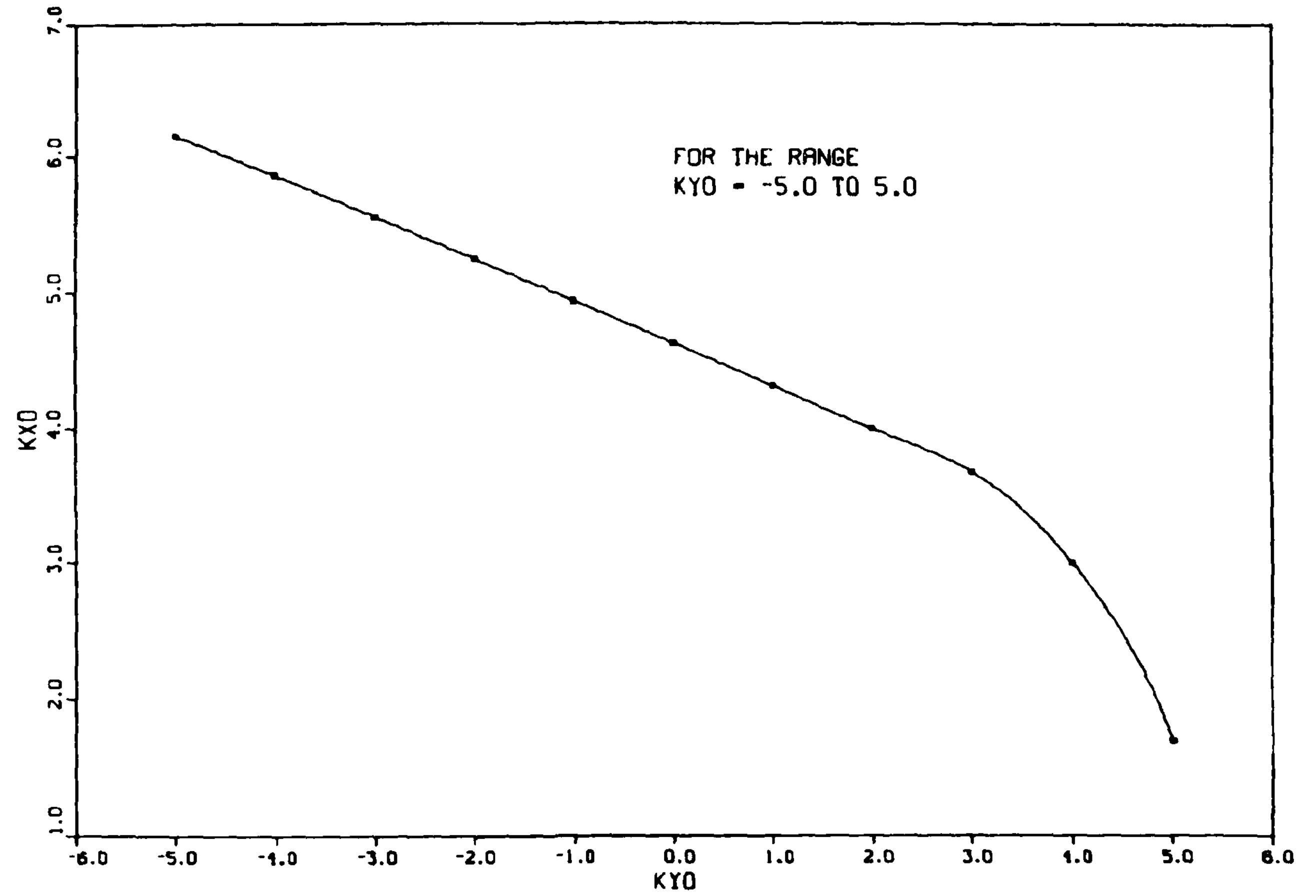

FIGURE 13

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.2 FOR AN S-C-S-C LAMINATE

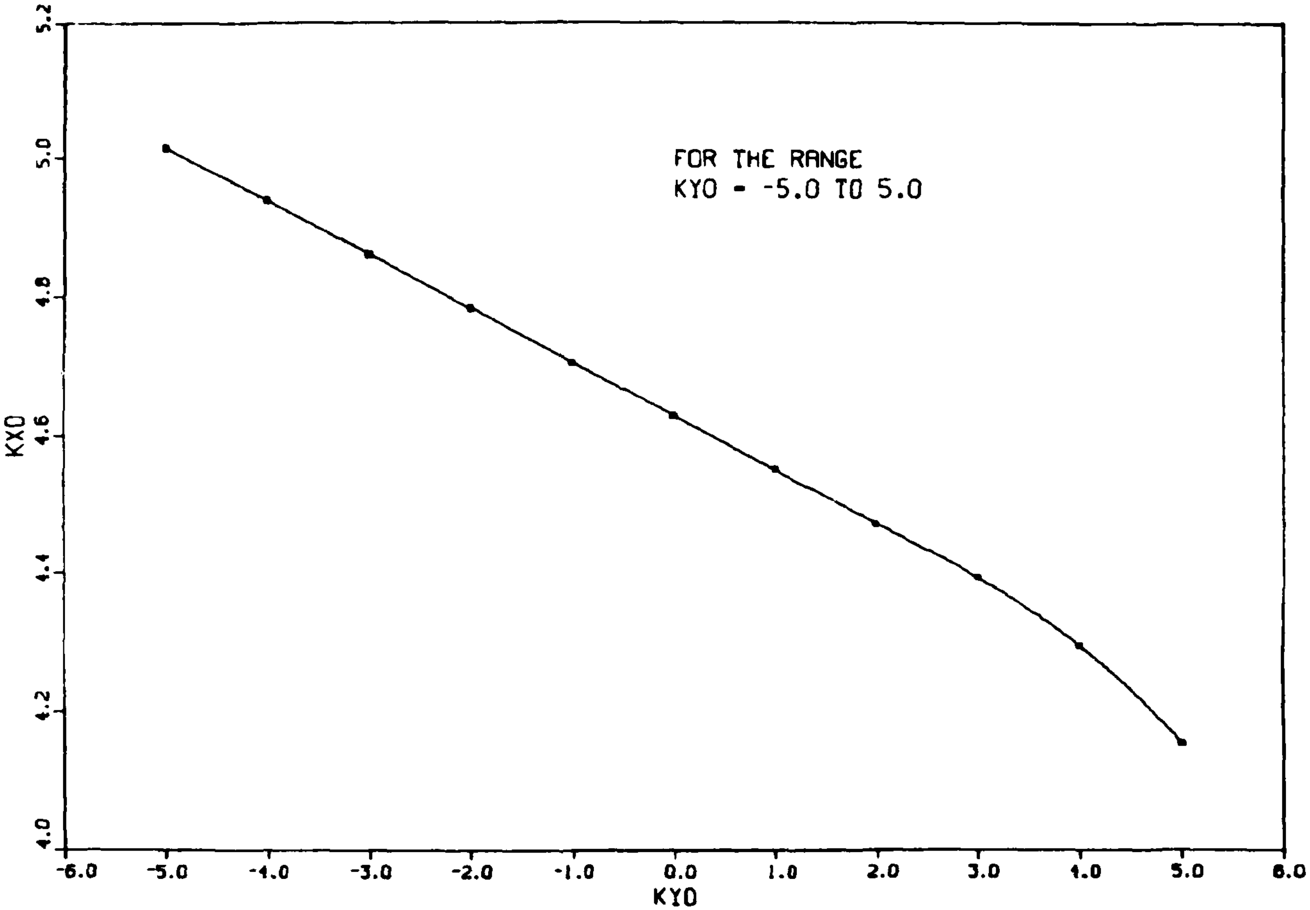

FIGURE 14

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.4 FOR AN S-C-S-C LAMINATE

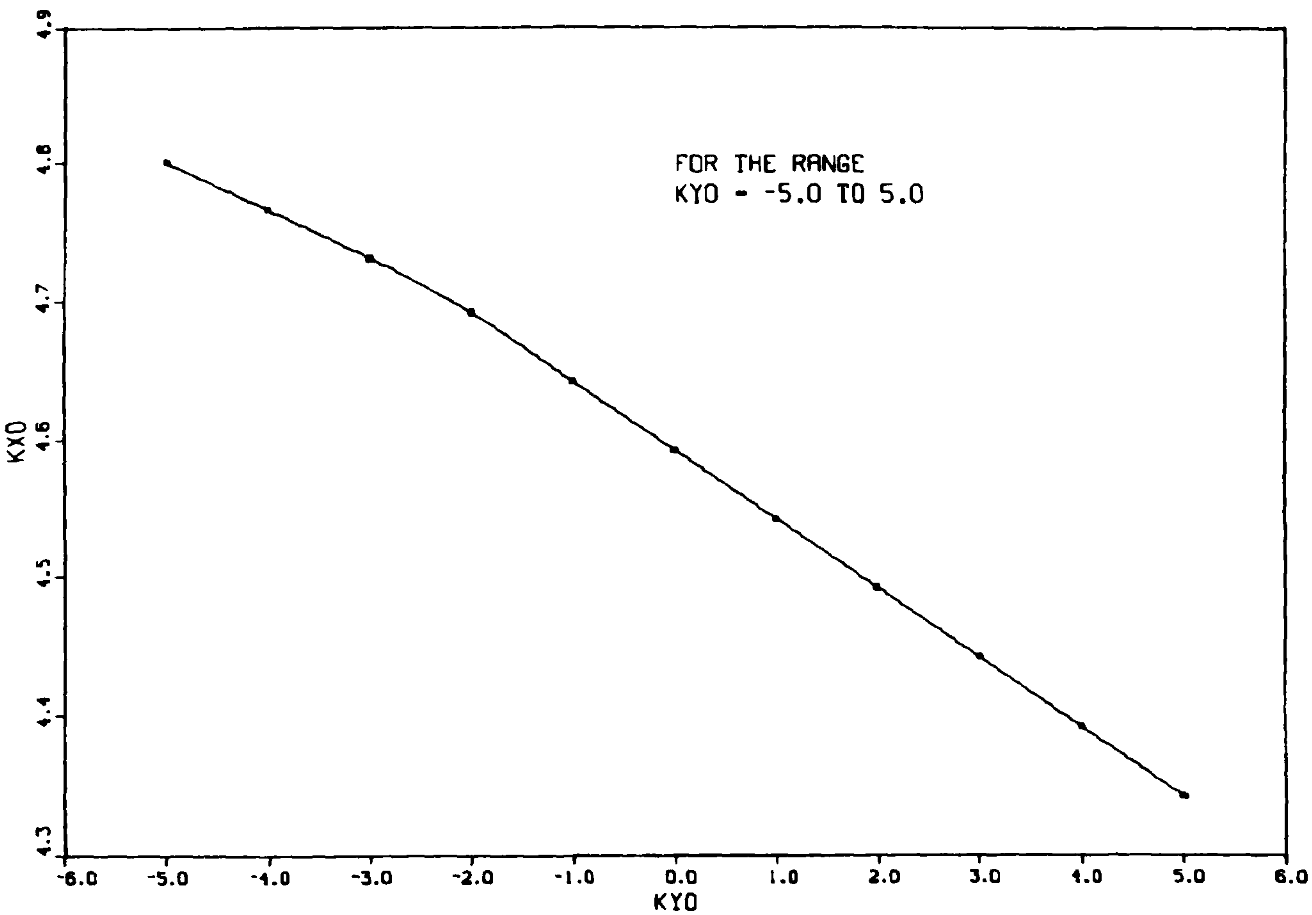

FIGURE 15

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 3.6 FOR AN S-C-S-C LAMINATE

V. Flat Rectangular Composite Laminate Simply Supported in the x_o-Direction and Simply Supported and Clamped on the Two Edges Normal to the y_o-Direction

The boundary conditions for a laminate simply supported in the x_o-direction and simply supported and clamped on the two edges perpendicular to the y_o-direction display no symmetry in the y_o-direction. For the two edges which have normals parallel to the x_o-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. Similarly, for that edge normal to the y_o-direction which is simply supported, these same edge conditions also hold. However, for the remaining edge, which is oriented perpendicular to the y_o-direction, the vertical displacement and the slope of the vertical displacement with respect to y_o must vanish. In equation form, the following must hold:

$$\begin{aligned}
&\text{on edge } x_o=-a_o/2\,, && w = 0\;; && w_{,x_o x_o} = 0\\
&\text{on edge } x_o= a_o/2\,, && w = 0\;; && w_{,x_o x_o} = 0\\
&\text{on edge } y_o= 0\,, && w = 0\;; && w_{,y_o y_o} = 0 \qquad (183)\\
&\text{on edge } y_o= b_o\,, && w = 0\;; && w_{,y_o} = 0
\end{aligned}$$

Note that the origin of coordinates in the affine space is taken to be at the center of the simply supported edge normal to the y_o-direction. This choice of origin location, in general, allows maximum simplicity in manipulations since a lack of symmetry is present in the boundary conditions in

the y_o-direction.

Just as before, a displacement function w which satisfies the first two stipulations of equations (183) is given by equation (77). Furthermore, substitution of this relation for w into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable r is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states--negative, zero, and positive--of the quantity in the square brackets of equation (81). Again, as explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

$\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$

For the quantity $\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$ equations (82) through (105) explicitly show that the unknown function $Y(y_o)$ must take the form shown in equation (106).

Consider now the boundary conditions, equations (183), for this case of a laminate simply supported on three sides and clamped on the fourth. When the chosen form of w, equation (77), is substituted into the final two equations of equations (183), the following must hold:

$$Y(0)\sin(m\pi x_o/a_o) = 0 \quad ; \quad Y''(0)\sin(m\pi x_o/a_o) = 0 \qquad (184)$$

$$Y(b_o)\sin(m\pi x_o/a_o) = 0 \quad ; \quad Y'(b_o)\sin(m\pi x_o/a_o) = 0 \qquad (185)$$

For equations (184) and (185) to have meaning in the general case, the following conditions must hold:

$$Y(0) = 0 \tag{186}$$

$$Y''(0) = 0 \tag{187}$$

$$Y(b_o) = 0 \tag{188}$$

$$Y'(b_o) = 0 \tag{189}$$

First, apply equation (186) to equation (106). This solution fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (187) on the $Y(y_o)$ equation similarly determines a value C_m in terms of A_m.

$$A_m(2cs) + B_m(c^2 - s^2) - C_m(2cs) + D_m(c^2 - s^2) = 0 \tag{190}$$

But since $D_m = -B_m$,

$$C_m = A_m(2cs)/(2cs) = A_m \tag{191}$$

For these values of C_m and D_m, equation (106) takes on the following form:

$$\begin{aligned} Y(y_o) = {} & A_m\{ \sin(sy_o)\ (e^{cy_o} + e^{-cy_o})\} \\ & + B_m\{ \cos(sy_o)\ (e^{cy_o} - e^{-cy_o})\} \end{aligned} \tag{192}$$

The first derivative of $Y(y_o)$ with respect to y_o is easy to obtain.

$$\begin{aligned} Y'(y_o) = {} & A_m\{ \cos(sy_o)\ (se^{cy_o} + se^{-cy_o}) \\ & \quad + \sin(sy_o)\ (ce^{cy_o} - ce^{-cy_o})\} \\ & + B_m\{ \sin(sy_o)\ (-se^{cy_o} + se^{-cy_o}) \\ & \quad + \cos(sy_o)\ (ce^{cy_o} + ce^{-cy_o})\} \end{aligned} \tag{193}$$

Substitution of equations (192) and (193) into equations (188) and (189) yield two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\left|\begin{array}{c|c} \sin(sb_o)(e^{cb_o}+e^{-cb_o}) & \cos(sb_o)(e^{cb_o}-e^{-cb_o}) \\ \hline \sin(sb\ )(ce^{cb_o}-ce^{-cb_o}) & \sin(sb_o)(-se^{cb_o}+se^{-cb_o}) \\ +\cos(sb_o)(se^{cb_o}+se^{-cb_o}) & +\cos(sb_o)(ce^{cb_o}+ce^{-cb_o}) \end{array}\right| = 0 \qquad (194)$$

Expansion of the determinant gives:

$$e^{2cb_o} - e^{-2cb_o} - 4(cb_o/sb_o)\sin(sb_o)\cos(sb_o) = 0 \qquad (195)$$

Unfortunately, for any combination of a_o/b_o and K_{y_o}, no value of K_{x_o} satisfies equation (195). In other words, no possible solutions exist for the present set of boundary conditions for $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of K_{y_o}, each possible range of K_{y_o} --negative, zero, and positive--will be analyzed as different subcases.

$K_{y_o} < 0$.

For $K_{y_o} < 0$ equations (121) and (122) explicitly demonstrate that the unknown function $Y(y_o)$ must take the form shown in equation (123). Equations (186) through (189) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (186) on equation (123) dictates that B_m must vanish. In addition, application

of equation (187) leads one to the conclusion that C_m is zero. So equation (123) takes on the following form:

$$Y(y_o) = A_m \sinh(Ty_o) + D_m y_o \cosh(Ty_o) \tag{196}$$

The first derivative of $Y(y_o)$ with respect to y_o is theretore easy to obtain.

$$Y'(y_o) = A_m T\cosh(Ty_o) + D_m\{\cosh(Ty_o) + Ty_o \sinh(Ty_o)\} \tag{197}$$

Substitution of equations (196) and (197) into equations (188) and (189) yield two homogeneous linear equations in coefficients A_m and D_m. For non-trivial A_m and D_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(Tb_o) & b_o\cosh(Tb_o) \\ T\cosh(Tb_o) & \cosh(Tb_o) + Tb_o\sinh(Tb_o) \end{vmatrix} = 0 \tag{198}$$

Expansion of the determinant gives:

$$\sinh(Tb_o)\cosh(Tb_o) - Tb_o = 0 \tag{199}$$

No value of Tb_o greater than zero can satisfy equation (199). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} < 0$

$K_{y_o} = 0$.

For $K_{y_o} = 0$ equation (128) constitutes the required shape of the unknown $Y(y_o)$ function. In addition, equations (186) through (189) are the sets of constraints for this $Y(y_o)$ function. Equations (186) and (187) imply that $A_m = C_m = 0$ Equations (188) and (189), with each side of (189) multiplied by b_o, yield the following set of simultaneous

equations:

$$B_m b_o + D_m b_o^3 = 0$$

$$B_m b_o + 3D_m b_o^3 = 0 \tag{200}$$

Only $B_m = D_m = 0$ constitutes a valid solution for equations (200). So again no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} = 0$

$K_{y_o} > 0.$

For $K_{y_o} > 0$ equations (130) and (131) explicitly demonstrate that the unknown function $Y(y_o)$ must take the form shown in equation (132). Equations (186) through (189) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (186) on equation (132) dictates that B_m must vanish. In addition, application of equation (187) leads one to the conclusion that C_m is zero. So equation (132) takes on the following form:

$$Y(y_o) = A_m \sin(Uy_o) + D_m y_o \cos(Uy_o) \tag{201}$$

The first derivative of $Y(y_o)$ with respect to y_o is theretore easy to obtain.

$$Y'(y_o) = A_m U\cos(Uy_o) + D_m\{\cos(Uy_o) - Uy_o \sin(Uy_o)\} \tag{202}$$

Substitution of equations (201) and (202) into equations (188) and (189) yields two homogeneous linear equations in coefficients A_m and D_m. For non-trivial A_m and D_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(Ub_o) & b_o\cos(Ub_o) \\ U\cos(Ub_o) & \cos(Ub_o) - Ub_o\sin(Ub_o) \end{vmatrix} = 0 \qquad (203)$$

Expansion of the determinant gives:

$$\sin(Ub_o)\cos(Ub_o) - Ub_o = 0 \qquad (204)$$

No value of Ub_o greater than zero can satisfy equation (204). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} > 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $Y(y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of K_{y_o} for valid solutions. Three subcases must be considered so that a solution for K_{x_o} may be determined for any range of K_{y_o}.

K_{y_o} <u>Ranges from a Relatively Large Positive Number to Positive Infinity</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity

likewise bracketed in equation (138) cannot be positive, K_{y_o} can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (186) through (189) again comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (186) to equation (144). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (187) on the $Y(y_o)$ equation, on the other hand, forces B_m and hence D_m to vanish. So equation (144) takes on the following form:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + C_m \sin(\nu_m y_o) \qquad (205)$$

The first derivative of $Y(y_o)$ with respect to y_o is easy to obtain.

$$Y'(y_o) = A_m \alpha_m \cos(\alpha_m y_o) + C_m \nu_m \cos(\nu_m y_o) \qquad (206)$$

Substitution of equations (205) and (206) into equations (188) and (189) yields two homogeneous linear equations in the coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o) & \sin(\nu_m b_o) \\ \alpha_m \cos(\alpha_m b_o) & \nu_m \cos(\nu_m b_o) \end{vmatrix} = 0 \qquad (207)$$

AD-A152 958 BIAXIAL BUCKLING OF SPECIALLY ORTHOTROPIC SYMMETRIC COMPOSITE RECTANGULAR PLATES(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. J P MCFADDEN 2/3

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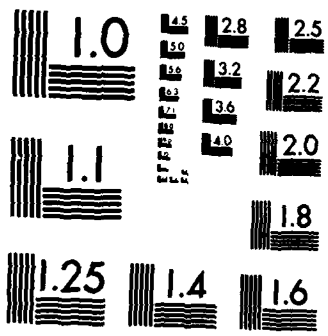

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Expansion of the determinant and multiplication by the quantity b_o gives:

$$\nu_m b_o \sin(\alpha_m b_o)\cos(\nu_m b_o) - \alpha_m b_o \sin(\nu_m b_o)\cos(\alpha_m b_o) = 0 \quad (208)$$

Attempts at solution of equation (208) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase are abandoned.

K_{y_o} Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, K_{y_o} can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that K_{y_o} range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (162). Note also that equations (145) and (163) define the variables in equation (162). Equations (186) through (189) once more comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (186) to equation (162). This stipulation fixes D_m in terms of B_m such that

$D_m = -B_m$ Utilization of equation (187) on the $Y(y_o)$ equation, on the other hand, forces B_m and hence D_m to vanish. So equation (162) takes on the following form:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + C_m \sinh(\beta_m y_o) \qquad (209)$$

The first derivative of $Y(y_o)$ with respect to y_o is easy to obtain.

$$Y'(y_o) = A_m \alpha_m \cos(\alpha_m y_o) + C_m \beta_m \cosh(\beta_m y_o) \qquad (210)$$

Substitution of equations (209) and (210) into equations (188) and (189) yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o) & \sinh(\beta_m b_o) \\ \alpha_m \cos(\alpha_m b_o) & \beta_m \cosh(\beta_m b_o) \end{vmatrix} = 0 \qquad (211)$$

Expansion of the determinant and multiplication by the quantity b_o gives:

$$\beta_m b_o \sin(\alpha_m b_o) \cosh(\beta_m b_o) - \alpha_m b_o \cos(\alpha_m b_o) \sinh(\beta_m b_o) = 0 \qquad (212)$$

Attempts at solution of equation (212) for any combination of a_o/b_o and K_{y_o} do yield the smallest values of K_{x_o}. Further consideration of results generated by equation (212) is postponed until the last of the three subcases is presented.

K_{y_o} <u>Ranges from a Relatively Large Negative Number to Negative Infinity</u>.

If the quantity contained in the curly brackets of

equation (137) is constrained to be greater than zero, K_{y_o} can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Moreover, equations (172), (173), and (174) sequentially illustrate that $Y(y_o)$ must take the form shown in equation (175). Note also that equations (163) and (176) define the variables in equation (175). Equations (186) through (189) once more comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (186) to equation (175). This combination fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (187) on the $Y(y_o)$ equation, in contrast, forces B_m and hence D_m to vanish. So equation (175) takes on the following form:

$$Y(y_o) = A_m \sinh(\theta_m y_o) + C_m \sinh(\beta_m y_o) \qquad (213)$$

The first derivative of $Y(y_o)$ with respect to y_o is easy to obtain.

$$Y'(y_o) = A_m \theta_m \cosh(\theta_m y_o) + C_m \beta_m \cosh(\beta_m y_o) \qquad (214)$$

Substitution of equations (213) and (214) into equations (188) and (189) yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(\theta_m b_o) & \sinh(\beta_m b_o) \\ \theta_m \cosh(\theta_m b_o) & \beta_m \cosh(\beta_m b_o) \end{vmatrix} = 0 \qquad (215)$$

Expansion of the determinant and multiplication by the quantity b_o gives:

$$\beta_m b_o \sinh(\theta_m b_o) \cosh(\beta_m b_o) - \theta_m b_o \sinh(\beta_m b_o) \cosh(\theta_m b_o) = 0 \qquad (216)$$

Attempts at solution of equation (216) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase as a whole are dropped.

Discussion of Results

Table VIII gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets as determined by equation (212). Also included is the integer value of m which produces this minimum K_{x_o}. Furthermore, each entry point which corresponds to a transition point from the m curve to the (m+1) curve is superscripted in the a_o/b_o column with a star (*).

The statistics presented in Table VIII expose two important characteristics of laminates under compression or tension in the y_o-direction. First, the transition values of a_o/b_o increase as K_{y_o} becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as a_o/b_o approaches a large number. Second, irrespective of the magnitude of K_{y_o}, K_{x_o} attains a limiting value of 3.125 as

a_o/b_o approaches infinity.

Figure 16 represents a plot of K_{x_o} versus a_o/b_o for eleven distinct values of K_{y_o} . The lowest curve characterizes $K_{y_o} = 5.0$; whereas, the highest depicts $K_{y_o} = -5.0$ The nine other curves differ from each other by increments of one. This graph reinforces the concept that K_{x_o} for a constant K_{y_o} is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of $K_{x_o} = 3.125$ for a_o/b_o large is readily apparent.

Figure 17 plots in three dimensions the same information as Figure 16. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 16; however, the quantitative aspect of Figure 17 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table IX gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o--1.6, 2.6, and 4.0 Figures 18, 19, and 20 represent two-dimensional plots at these constant a_o/b_o slices of 1.6, 2.6, and 4.0, respectively.

These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that K_{x_o} declines as K_{y_o} increases and that the rate of decline of K_{x_o} for an increase in K_{y_o} jumps markedly for small a_o/b_o.

TABLE VIII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o -Direction and Simply Supported and Clamped on the Two Edges Normal to the y_o -Direction

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	-3.0	7.0176
1	0.8714*	-3.0	6.5847
2	1.4000	-3.0	4.1016
2	1.7873*	-3.0	4.0696
3	2.2000	-3.0	3.5588
3	2.6458*	-3.0	3.5713
4	3.0000	-3.0	3.3686
4	3.4795*	-3.0	3.3864
5	3.8000	-3.0	3.2806
5	4.3013*	-3.0	3.2971
1	0.6000	0.0	3.6563
1	1.1315*	0.0	3.9055
2	1.4000	0.0	3.2366
2	1.9598*	0.0	3.3847
3	2.4000	0.0	3.1244
3	2.7716*	0.0	3.2546
4	3.2000	0.0	3.1244
4	3.5781*	0.0	3.2025
5	3.8000	0.0	3.1409
5	4.3822*	0.0	3.1765
1	1.4000	3.0	1.7388
1	1.4829*	3.0	2.2738
2	1.8000	3.0	2.3287
2	2.1515*	3.0	2.8083
3	2.4000	3.0	2.7337
3	2.9042*	3.0	2.9641
4	3.2000	3.0	2.9051
4	3.6798*	3.0	3.0278
5	4.2000	3.0	2.9990
5	4.4649*	3.0	3.0599

TABLE IX

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported in the x_o-Direction and Simply Supported and Clamped in the y_o-Direction

m	a_o/b_o	K_{y_o}	K_{x_o}
2	1.6000	-5.0	4.5621
2	1.6000	-3.0	3.9914
2	1.6000	-1.0	3.4151
2	1.6000	0.0	3.1244
2	1.6000	1.0	2.8317
2	1.6000	3.0	2.2239
1	1.6000	5.0	0.6812
4	2.6000	-5.0	3.7603
3	2.6000	-3.0	3.5518
3	2.6000	-1.0	3.2938
3	2.6000	0.0	3.1644
3	2.6000	1.0	3.0346
3	2.6000	3.0	2.7740
3	2.6000	5.0	2.5119
5	4.0000	-5.0	3.3571
5	4.0000	-3.0	3.2641
5	4.0000	-1.0	3.1710
5	4.0000	0.0	3.1244
5	4.0000	1.0	3.0777
5	4.0000	3.0	2.9842
5	4.0000	5.0	2.8904

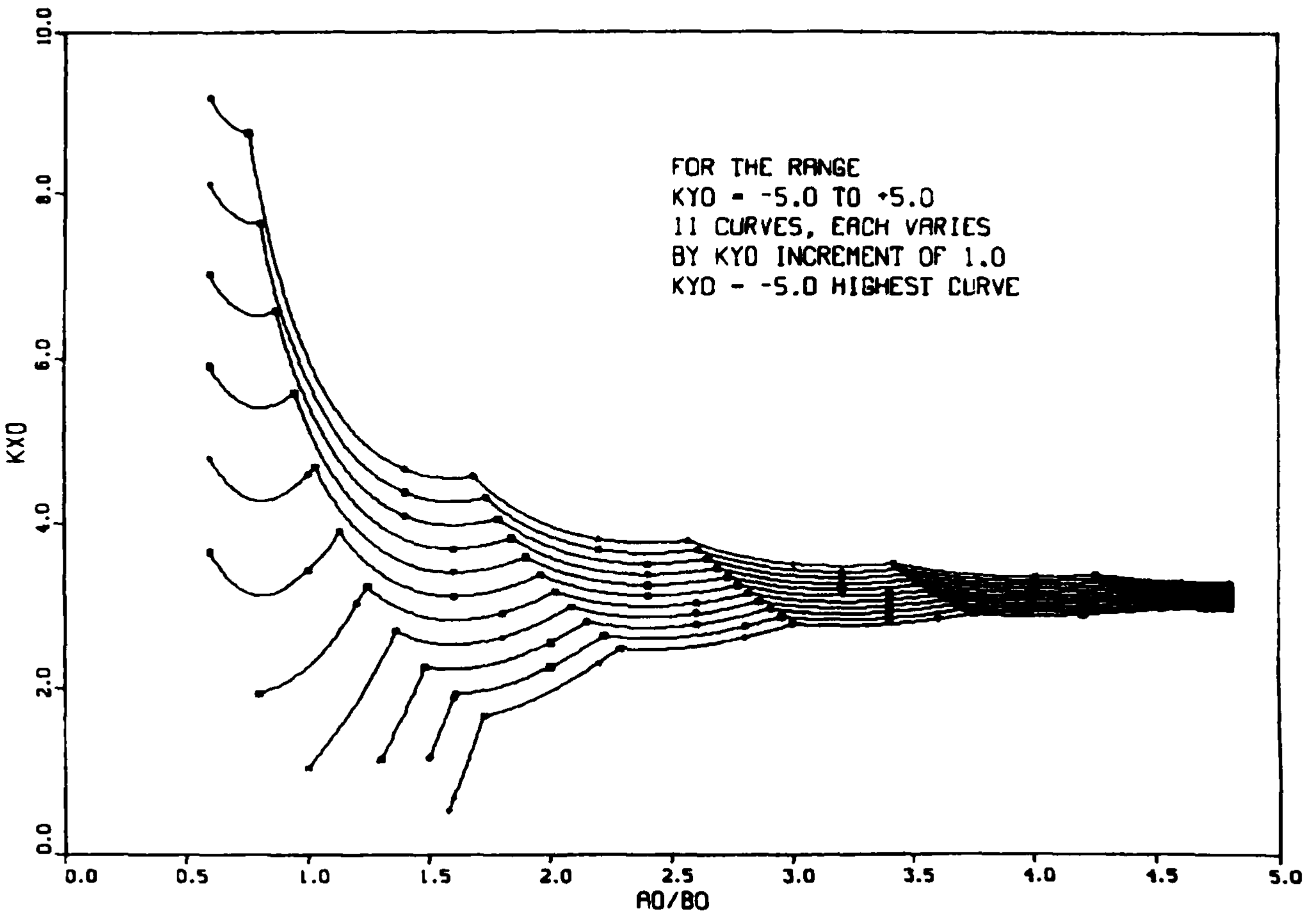


FIGURE 16

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-C-S-S LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

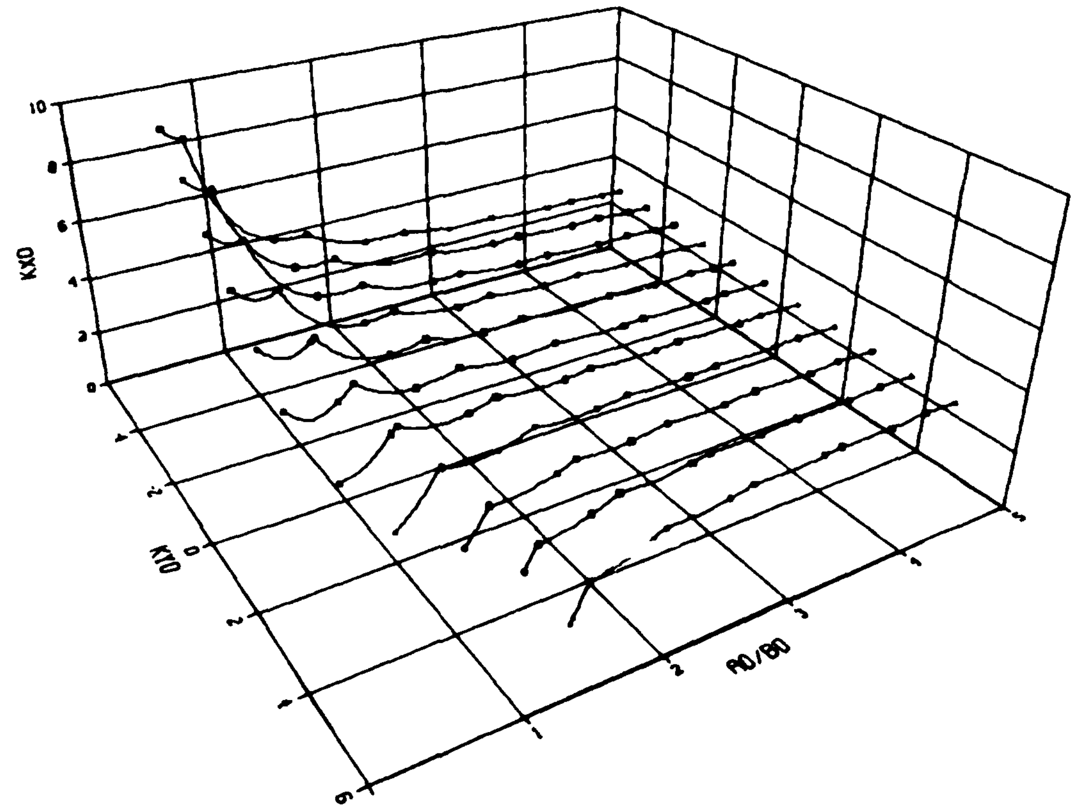


FIGURE 17

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO FOR AN S-C-S-S LAMINATE

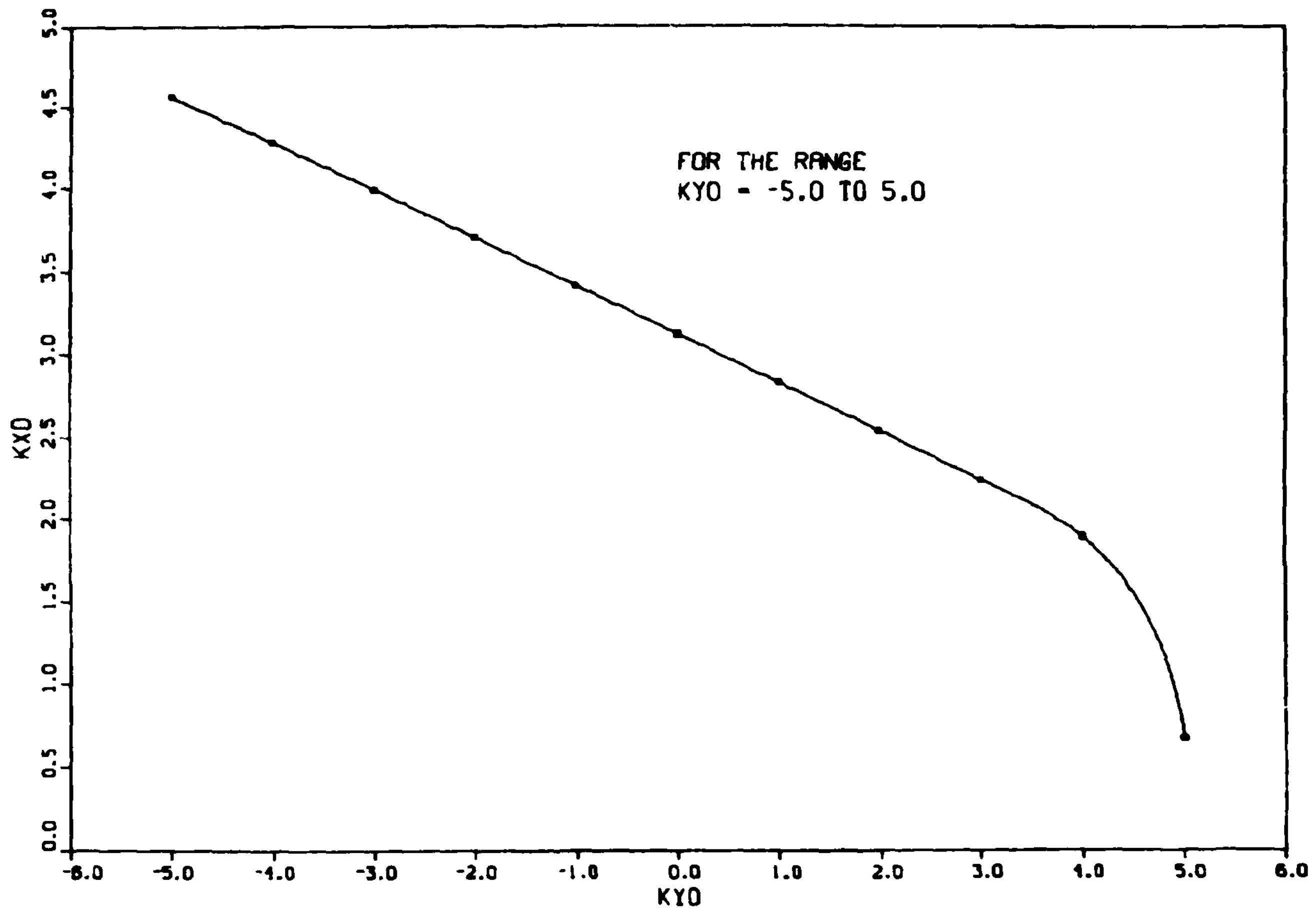


FIGURE 18

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.6 FOR AN S-C-S-S LAMINATE

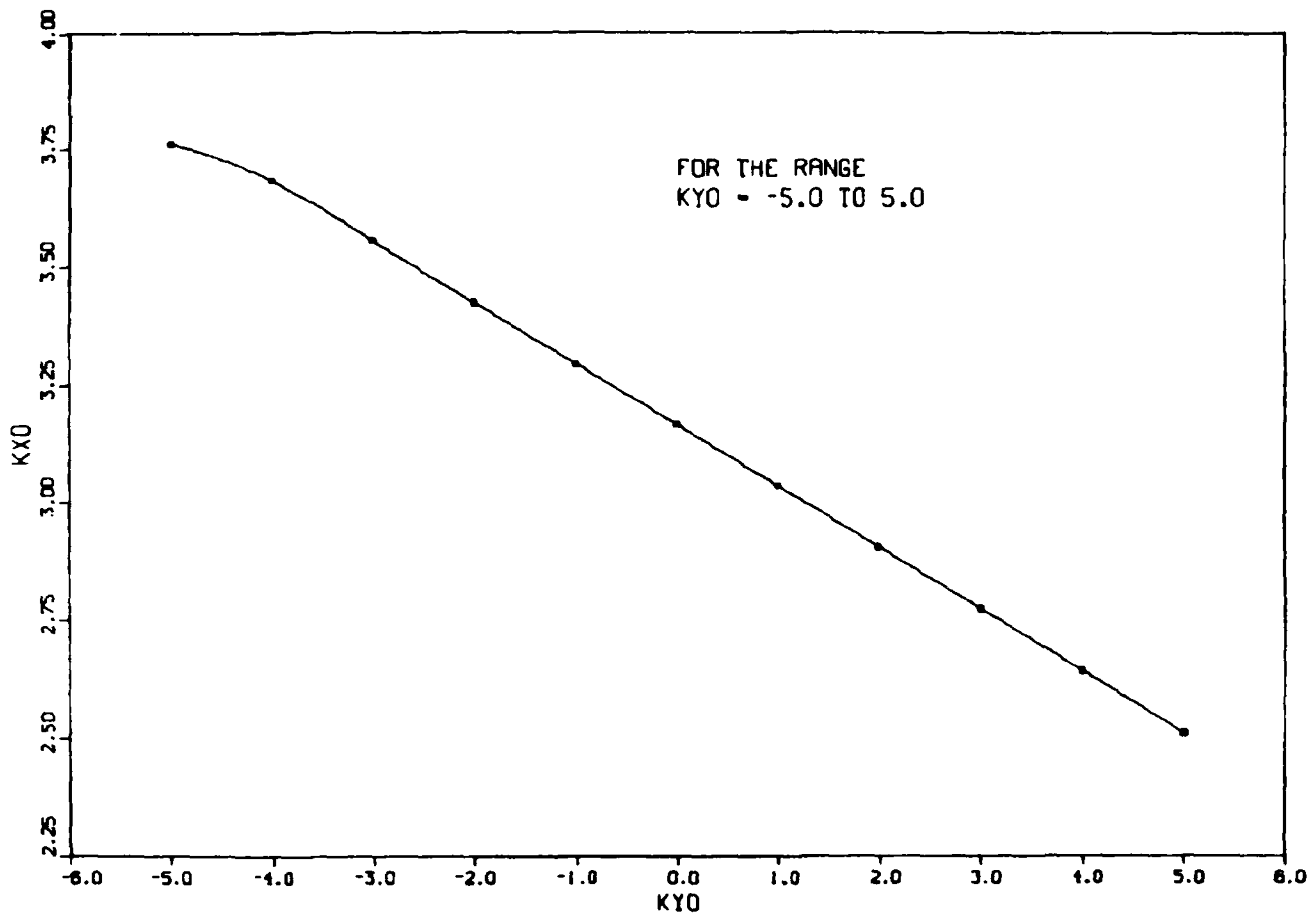


FIGURE 19

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.6 FOR AN S-C-S-S LAMINATE

can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, K_{y_o} can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (228) through (231) once more comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (228) to equation (144). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (229), on the other hand, yields a relation between C_m and A_m such that $C_m = -(\alpha_m/\gamma_m)A_m$ So equation (144) takes on the following form:

$$Y(y_o) = A_m\{\sin(\alpha_m y_o) - [\alpha_m/\gamma_m]\sin(\gamma_m y_o)\} + B_m\{\cos(\alpha_m y_o) - \cos(\gamma_m y_o)\} \quad (255)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{\alpha_m\cos(\alpha_m y_o) - \alpha_m\cos(\gamma_m y_o)\} - B_m\{\alpha_m\sin(\alpha_m y_o) - \gamma_m\sin(\gamma_m y_o)\} \quad (256)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m\{\alpha_m^2\sin(\alpha_m y_o) - \alpha_m\gamma_m\sin(\gamma_m y_o)\} - B_m\{\alpha_m^2\cos(\alpha_m y_o) - \gamma_m^2\cos(\gamma_m y_o)\} \quad (257)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o

quantity b_o^4 results in the following system of equations (the second equation a simplification of the first):

$$\sin^2(Ub_o)\{-(Ub_o)^6 - 3(Ub_o)^4 + K_{y_o}(\pi b_o/a_o)^2[(Ub_o)^4 + (Ub_o)^2]\}$$
$$+\cos^2(Ub_o)\{-(Ub_o)^6 - 4(Ub_o)^4 + K_{y_o}(\pi b_o/a_o)^2(Ub_o)^4\} = 0 \quad (253)$$

$$K_{y_o}(\pi b_o/a_o)^2\{(Ub_o)^4 + (Ub_o)^2\sin^2(Ub_o)\}$$
$$- (Ub_o)^4\{3 + \cos^2(Ub_o) + (Ub_o)^2\} = 0 \quad (254)$$

No value of (Ub_o) greater than zero can satisfy equation (254). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} > 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $Y(y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of K_{y_o} for valid solutions. Three subcases must be considered so that a solution for K_{x_o} may be determined for any range of K_{y_o}.

K_{y_o} Ranges from a Relatively Large Positive Number to Positive Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o}

Equation (228) implies that B_m must vanish, and equation (229) can be satisfied only if $D_m = -A_m U$ So equation (132) takes on the following form:

$$Y(y_o) = A_m\{\sin(Uy_o) - Uy_o\cos(Uy_o)\} + C_m y_o \sin(Uy_o) \qquad (248)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m U^2 y_o \sin(Uy_o) + C_m\{\sin(Uy_o) + Uy_o\cos(Uy_o)\} \qquad (249)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y''(y_o) &= A_m\{U^2\sin(Uy_o) + U^3 y_o\cos(Uy_o)\} \\ &\quad + C_m\{2U\cos(Uy_o) - U^2 y_o\sin(Uy_o)\} \end{aligned} \qquad (250)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y'''(y_o) &= A_m\{2U^3\cos(Uy_o) - U^4 y_o\sin(Uy_o)\} \\ &\quad + C_m\{-3U^2\sin(Uy_o) - U^3 y_o\cos(Uy_o)\} \end{aligned} \qquad (251)$$

Substitution of equations (249), (250), and (251) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(Ub_o)\,\{U^2\} + \cos(Ub_o)\,\{U^3 b_o\} & \sin(Ub_o)\,\{-U^2 b_o\} + \cos(Ub_o)\,\{2U\} \\ \sin(Ub_o)\,\{-U^4 b_o + K_{y_o}(\pi/a_o)^2 U^2 b_o\} + \cos(Ub_o)\,\{2U^3\} & \sin(Ub_o)\,\{-3U^2 + K_{y_o}(\pi/a_o)^2\} + \cos(Ub_o)\,\{-U^3 b_o + K_{y_o}(\pi/a_o)^2 U b_o\} \end{vmatrix} = 0 \qquad (252)$$

Expansion of the determinant and multiplication by the

$$\cosh^2(Tb_o)\ \{-(Tb_o)^6 + 4(Tb_o)^4 - K_{y_o}(\pi b_o/a_o)^2(Tb_o)^4\}$$
$$- \sinh^2(Tb_o)\ \{-(Tb_o)^6 + 3(Tb_o)^4$$
$$+ K_{y_o}(\pi b_o/a_o)^2[(Tb_o)^2 - (Tb_o)^4\ ]\ \} = 0 \quad (246)$$

$$(Tb_o)^4\ \{\ 3 + \cosh^2(Tb_o) - (Tb_o)^2\ \}$$
$$- K_{y_o}(\pi b_o/a_o)^2\ \{\ (Tb_o)^4 + (Tb_o)^2 \sinh^2(Tb_o)\ \} = 0 \quad (247)$$

No value of (Tb_o) greater than zero can satisfy equation (247). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} < 0$

$K_{y_o} = 0.$

For $K_{y_o} = 0$ equation (128) constitutes the required shape of the unknown $Y(y_o)$ function. In addition, equations (228) through (231) are the sets of constraints for this $Y(y_o)$ function. Moreover, notice that equation (231) reduces to $Y'''(b_o) = 0$ in this instance since $K_{y_o} = 0$ Equations (228) and (229) imply that $A_m = B_m = 0$ Also, satisfaction of equation (231) necessitates that the constant D_m must likewise have the null value. Finally, and in this sequence, enforcement of equation (230) reveals that C_m too must vanish. Thus, the function $Y(y_o)$ is nothing more than the trivial function for $K_{y_o} = 0$ and shall be ignored.

$K_{y_o} > 0.$

For $K_{y_o} > 0$ equations (130) and (131) show that the unknown function $Y(y_o)$ must fit the relation given by equation (132). Equations (228) through (231) again comprise the group of boundary conditions for the $Y(y_o)$ function.

$$Y'(y_o) = -A_m T^2 y_o \sinh(Ty_o) + C_m\{\sinh(Ty_o) + Ty_o\cosh(Ty_o)\} \qquad (242)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m\{-T^2\sinh(Ty_o) - T^3 y_o\cosh(Ty_o)\} + C_m\{2T\cosh(Ty_o) + T^2 y_o\sinh(Ty_o)\} \qquad (243)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{-2T^3\cosh(Ty_o) - T^4 y_o\sinh(Ty_o)\} + C_m\{3T^2\sinh(Ty_o) + T^3 y_o\cosh(Ty_o)\} \qquad (244)$$

Substitution of equations (242), (243), and (244) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determ'nental equation must hold:

$$\begin{vmatrix} \sinh(Tb_o)\,\{-T^2\} - \cosh(Tb_o)\,\{T^3 b_o\} & \sinh(Tb_o)\,\{T^2 b_o\} + \cosh(Tb_o)\,\{2T\} \\ \sinh(Tb_o)\,\{-T^4 b_o - K_{y_o}(\pi/a_o)^2 T^2 b_o\} - \cosh(Tb_o)\,\{2T^3\} & \sinh(Tb_o)\,\{3T^2 + K_{y_o}(\pi/a_o)^2\} + \cosh(Tb_o)\,\{T^3 b_o + K_{y_o}(\pi/a_o)^2 Tb_o\} \end{vmatrix} = 0 \qquad (245)$$

Expansion of the determinant and multiplication by the quantity b_o^4 results in the following equations (the second equation a simplification of the first):

determination of minimum K_{x_o} for the combination of any plate aspect ratio and K_{y_o} greater than zero. In other words, the roots of equation (240) yield the smallest values of K_{x_o} for any a_o/b_o and compressive K_{y_o}. Consideration of results generated by equation (240) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically with the choice of algebraic sign of K_{y_o}, each possible range of K_{y_o} --negative, zero, and positive--will be analyzed as different subcases.

$K_{y_o} < 0$.

For $K_{y_o} < 0$ equations (121) and (122) lead to the conclusion that the unknown function $Y(y_o)$ must take the form shown in equation (123). Equations (228) through (231) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (228) on equation (123) dictates that B_m must vanish. Application of equation (229), on the other hand, gives the relation that $D_m = -A_mT$ Therefore, equation (123) takes on the following form:

$$Y(y_o) = A_m\{\sinh(Ty_o) - Ty_o\cosh(Ty_o)\} + C_m y_o \sinh(Ty_o) \quad (241)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

quantity b_o^5 results in the following equations (the second equation a simplification of the first):

$$\begin{aligned}
&\sin^2(sb_o)\ [e^{2cb_o}\{-(sb_o)^5 - 2(cb_o)^2(sb_o)^3 - (cb_o)^4(sb_o) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2((cb_o)^2(sb_o) + (sb_o)^3)\} \\
&\qquad +e^{-2cb_o}\{-(sb_o)^5 - 2(cb_o)^2(sb_o)^3 - (cb_o)^4(sb_o) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2((cb_o)^2(sb_o) + (sb_o)^3)\} \\
&\qquad + \{-8(cb_o)^2(sb_o)^3 - 6(cb_o)^4(sb_o) + 2(sb_o)^5 \\
&\qquad + 4(cb_o)^6/(sb_o) + K_{y_o}(\pi b_o/a_o)^2(4(cb_o)^4/(sb_o) \\
&\qquad + 2(cb_o)^2(sb_o) - 2(sb_o)^3)\}\ ] \\
&+\cos^2(sb_o)[e^{2cb_o}\{-(sb_o)^5 - 2(cb_o)^2(sb_o)^3 - (cb_o)^4(sb_o) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2((cb_o)^2(sb_o) + (sb_o)^3)\} \\
&\qquad +e^{-2cb_o}\{-(sb_o)^5 - 2(cb_o)^2(sb_o)^3 - (cb_o)^4(sb_o) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2((cb_o)^2(sb_o) + (sb_o)^3)\} \\
&\qquad + \{-12(cb_o)^2(sb_o)^3 - 14(cb_o)^4(sb_o) + 2(sb_o)^5 \\
&\qquad - K_{y_o}(\pi b_o/a_o)^2(2(sb_o)^3 \\
&\qquad + 2(cb_o)^2(sb_o))\}\ ] = 0 \qquad (239)
\end{aligned}$$

$$\begin{aligned}
&(e^{2cb_o} + e^{-2cb_o})\{-(sb_o)^5 - 2(cb_o)^2(sb_o)^3 - (cb_o)^4(sb_o) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2((cb_o)^2(sb_o) + (sb_o)^3)\} \\
&\quad - \{8(cb_o)^2(sb_o)^3 + 6(cb_o)^4(sb_o) - 2(sb_o)^5 \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2(2(sb_o)^3)\} \\
&\quad - \cos^2(sb_o)\{4(cb_o)^2(sb_o)^3 + 8(cb_o)^4(sb_o) \qquad (240) \\
&\qquad + K_{y_o}(\pi b_o/a_o)^2(2(cb_o)^2(sb_o))\} \\
&\quad + \sin^2(sb_o)\{4(cb_o)^6/(sb_o) \\
&\quad + K_{y_o}(\pi b_o/a_o)^2[4(cb_o)^4/(sb_o) + 2(cb_o)^2(sb_o)]\} = 0
\end{aligned}$$

Equation (240) constitutes the governing equation for

$$
\begin{aligned}
Y'''(y_o) = A_m\{ & [(3c^2s - s^3)e^{cy_o} + (s^3 - 3c^2s)e^{-cy_o}]\cos(sy_o) \\
& + [(c^3 - 3cs^2)e^{cy_o} + (c^3 - 3cs^2)e^{-cy_o}]\sin(sy_o)\} \\
+ B_m\{ & [(c^3 - 3cs^2)e^{cy_o} - (5c^3 + cs^2)e^{-cy_o}]\cos(sy_o) \\
& + [(s^3 - 3c^2s)e^{cy_o} + (2(c^4/s) - 3c^2s \\
& - s^3)e^{-cy_o}]\sin(sy_o)\} \qquad (237)
\end{aligned}
$$

Substitution of equations (235), (236), and (237) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$
\begin{vmatrix}
\begin{array}{l}
\sin(sb_o)\ \{(c^2-s^2)e^{cb_o} \\
\quad +(s^2-c^2)e^{-cb_o}\} \\
+\cos(sb_o)\ (2cs)\ \{e^{cb_o} \\
\quad +e^{-cb_o}\}
\end{array}
&
\begin{array}{l}
\sin(sb_o)\ \{-2cse^{cb_o} \\
\quad -2(c^3/s)e^{-cb_o}\} \\
+\cos(sb_o)\ \{(c^2-s^2)e^{cb_o} \\
\quad +(3c^2+s^2)e^{-cb_o}\}
\end{array}
\\
\hline
\begin{array}{l}
\sin(sb_o)\ \{(c^3-3cs^2)e^{cb_o} \\
\quad +(c^3-3cs^2)e^{-cb_o} \\
+K_{y_o}(\pi/a_o)^2\ [ce^{cb_o} \\
\quad +ce^{-cb_o}]\} \\
\\
+\cos(sb_o)\ \{(3c^2s-s^3)e^{cb_o} \\
\quad +(s^3-3c^2s)e^{-cb_o} \\
+K_{y_o}(\pi/a_o)^2\ [se^{cb_o} \\
\quad -se^{-cb_o}]\}
\end{array}
&
\begin{array}{l}
\sin(sb\ )\ \{(s^3-3c^2s)e^{cb} \\
\quad +(2(c^4/s) - s^3 \\
\quad -3c^2s)e^{-cb} \\
+K_{y_o}(\pi/a_o)^2\ [(2c^2/s \\
\quad +s)e^{-cb_o} - se^{cb_o}]\} \\
+\cos(sb_o)\ \{(c^3-3cs^2)e^{cb_o} \\
\quad -(5c^3+cs^2)e^{-cb_o} \\
+K_{y_o}(\pi/a_o)^2\ [ce^{cb_o} \\
\quad -ce^{-cb_o}]\}
\end{array}
\end{vmatrix}
= 0 \qquad (238)
$$

Expansion of the determinant and multiplication by the

First, apply equation (228) to equation (106). This solution fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (229) on the $Y(y_o)$ equation similarly determines a value C_m in terms of A_m and B_m.

$$sA_m + cB_m + sC_m - cD_m = 0 \qquad (232)$$

But since $D_m = -B_m$

$$C_m = -A_m - 2\,B_m(c/s) \qquad (233)$$

For these values of C_m and D_m , equation (106) takes on the following form:

$$Y(y_o) = A_m(e^{cy_o} - e^{-cy_o})\sin(sy_o) + B_m\{(e^{cy_o} - e^{-cy_o})\cos(sy_o) - 2\,(c/s)\,e^{-cy_o}\sin(sy_o)\} \qquad (234)$$

The first derivative of $Y(y_o)$ with respect to y_o is therefore:

$$\begin{aligned} Y'(y_o) = {} & A_m\{(se^{cy_o} - se^{-cy_o})\cos(sy_o) \\ & +(ce^{cy_o} + ce^{-cy_o})\sin(sy_o)\} \\ & + B_m\{(ce^{cy_o} - ce^{-cy_o})\cos(sy_o) \\ & +(-se^{cy_o} + se^{-cy_o} + 2\,(c^2/s)e^{-cy_o})\sin(sy_o)\} \end{aligned} \qquad (235)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y''(y_o) = {} & A_m\{(2cse^{cy_o} + 2cse^{-cy_o})\cos(sy_o) \\ & +[(c^2 - s^2)e^{cy_o} + (s^2 - c^2)e^{-cy_o}]\sin(sy_o)\} \\ & + B_m\{[(c^2 - s^2)e^{cy_o} + (3c^2 + s^2)e^{-cy_o}]\cos(sy_o) \\ & +[-2cse^{cy_o} - 2(c^3/s)e^{-cy_o}]\sin(sy_o)\} \end{aligned} \qquad (236)$$

The third derivative of $Y(y_o)$ with respect to y_o is:

equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states--negative, zero, and positive--of the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

$\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$

For the quantity $\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} < 0$ equations (82) through (105) illustrate that the unknown function $Y(y_o)$ must take the form shown in equation (106).

Consider now the boundary conditions, equations (225), for this case of a laminate simply supported on one pair of opposite sides, clamped on a third, and free on the fourth. When the chosen form of w, equation (77), is substituted into the final three lines of equations (225), the following must hold:

$$Y(0) \sin(m\pi x_o/a_o) = 0 \;; \quad Y'(0) \sin(m\pi x_o/a_o) = 0 \qquad (226)$$

$$Y''(b_o) \sin(m\pi x_o/a_o) = 0 \;;$$

$$[Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o)] \sin(m\pi x_o/a_o) = 0 \qquad (227)$$

For equations (226) and (227) to have meaning in the general case, the following conditions must hold:

$$Y(0) = 0 \qquad (228)$$

$$Y'(0) = 0 \qquad (229)$$

$$Y''(b_o) = 0 \qquad (230)$$

$$Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o) = 0 \qquad (231)$$

equation (11)). The key assumption of this investigation is that $D^* = 0$ and this provision will continue to be enforced. Consequently, equation (223) reduces to the relatively simple result:

$$w_{,y_o y_o y_o} + K_{y_o} (\pi/a_o)^2 w_{,y_o} = 0 \qquad (224)$$

Equation (224) is the most convenient means to express the second boundary condition for a free edge normal to the y_o-direction.

As a recap and in equation form, the following are the boundary conditions for each of the four edges:

$$\begin{aligned}
&\text{on edge } x_o = -a_o/2 \,, \quad w = 0 \;;\; w_{,x_o x_o} = 0 \\
&\text{on edge } x_o = a_o/2 \,, \quad w = 0 \;;\; w_{,x_o x_o} = 0 \\
&\text{on edge } y_o = 0 \,, \quad w = 0 \;;\; w_{,y_o} = 0 \qquad (225) \\
&\text{on edge } y_o = b_o \,, \quad w_{,y_o y_o} = 0 \;; \\
&\qquad w_{,y_o y_o y_o} + K_{y_o} (\pi/a_o)^2 w_{,y_o} = 0
\end{aligned}$$

Note that the origin of coordinates in the affine space is taken to be at the center of the clamped edge normal to the y_o -direction. This choice of origin location, in general, allows maximum simplicity in manipulations for a lack of symmetry is present in the boundary conditions in the y_o-direction.

Just as before, a displacement function w which satisfies the first two stipulations of equations (225) is given by equation (77). Furthermore, substitution of this relation for w into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable r is defined in

Note that this boundary condition (217) is expressed in real space coordinates. No conversion to affine space values has yet been made. Q_y can be expressed in terms of the moments M_y and M_{xy} defined in equation (2) (3:326-327).

$$Q_y = -M_{y,y} - M_{xy,y} \qquad (218)$$

Appropriate partial differentiation of equation (2) gives the following identities (for symmetric, specially orthotropic laminates):

$$M_{y,y} = -D_{12} w_{,xxy} - D_{22} w_{,yyy} \qquad (219)$$

$$M_{xy,y} = -2\, D_{66} w_{,xxy} \qquad (220)$$

Replacement of terms in equation (217) by those identities given in equations (218), (219), and (220) yields:

$$D_{12} w_{,xxy} + D_{22} w_{,yyy} + 2\, D_{66} w_{,xxy} - N_y w_{,y} = 0 \qquad (221)$$

Now, transformation of these real space coordinates into affine space coordinates by the rules of equations (8) and utilization of the definition given by equation (13) reshapes equation (221) in the following way:

$$[D_{12}/(A^2B)]\, w_{,x_o x_o y_o} + [D_{22}/(B^3)]\, w_{,y_o y_o y_o}$$
$$+ [2\, D_{66}/(A^2B)]\, w_{,x_o x_o y_o} + [D_{22}^{1/2}/B]\, K_{y_o} (\pi/a_o)^2 w_{,y_o} = 0 \qquad (222)$$

The constants A and B were defined in the first section in terms of D_{11} and D_{22} such that $A = D_{11}^{1/4}$ and $B = D_{22}^{1/4}$ Application of these bits of information and division of both sides of equation (222) by $D_{22}^{1/4}/2$ gives:

$$[2(D_{12} + 2\, D_{66})/(D_{11} D_{22})^{1/2}]\, w_{,x_o x_o y_o}$$
$$+ 2\, w_{,y_o y_o y_o} + 2\, K_{y_o} (\pi/a_o)^2 w_{,y_o} = 0 \qquad (223)$$

Recognize that the coefficient of the first term on the left-hand side of equation (223) is merely D^* (defined in

VI. Flat Rectangular Composite Laminate Simply Supported in the x_0-Direction and Clamped and Free on the Two Edges Normal to the y_0-Direction

The boundary conditions for a laminate simply supported in the x_0-direction and clamped and free on the two edges perpendicular to the y_0-direction again display no symmetry in the y_0-direction. For the two edges which have normals parallel to the x_0-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. For the clamped edge normal to the y_0-direction, the vertical displacement and the slope of the vertical displacement with respect to y_0 must each equal zero. For the free edge normal to the y_0-direction, a moment's reflection allows one to formulate the appropriate boundary conditions.

First, like the simply supported edge, the normal component of the moment to the free edge must be zero. Second, the free edge can support no force on its boundary, so the summation of the shear force and the y -buckling load in the direction of the shear must equal zero.

$$Q_y - N_y w_{,y} = 0 \qquad (217)$$

where

Q_y = resultant shear force per unit length

N_y = normal force per unit length in the y-direction positive in tension

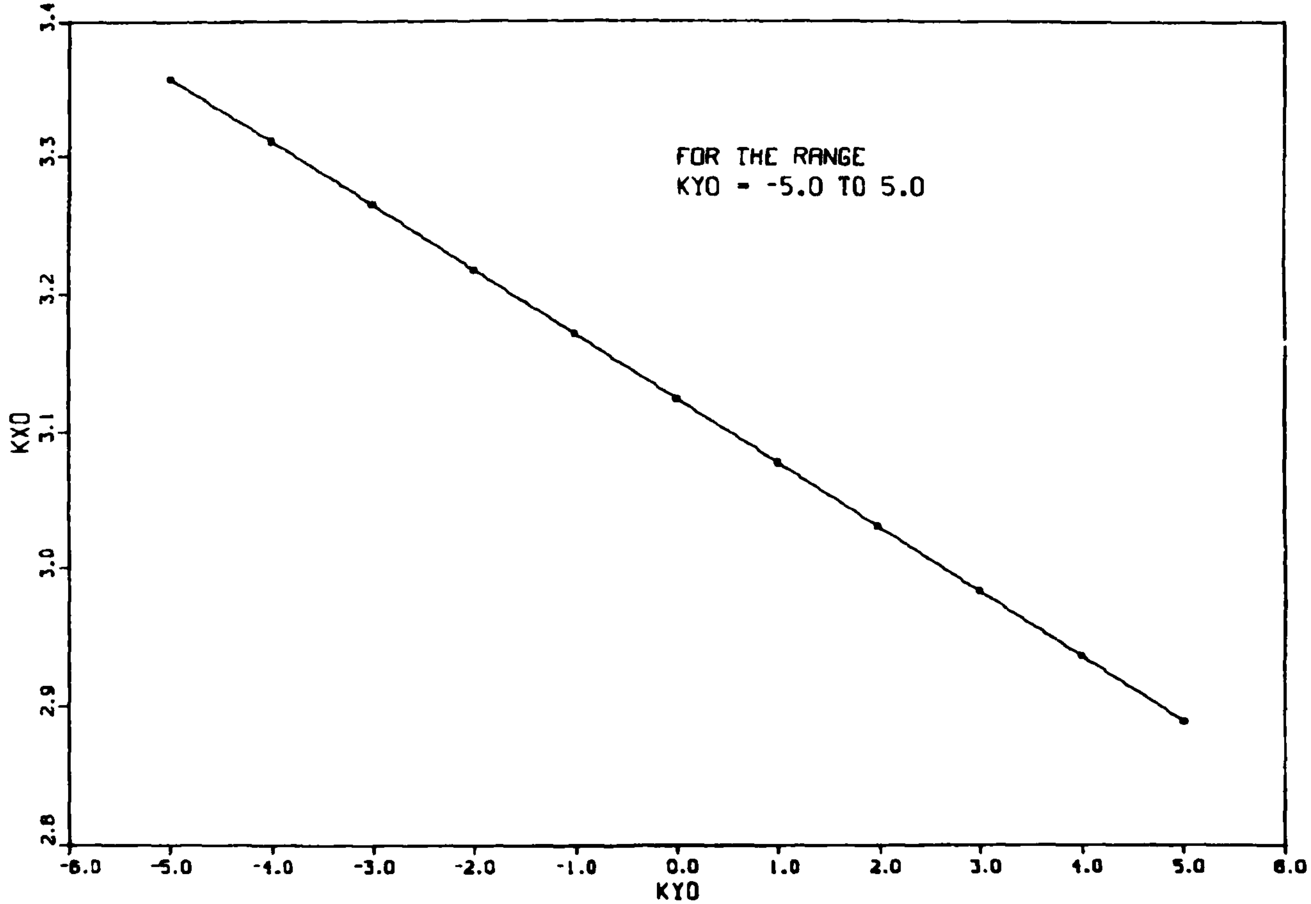


FIGURE 20

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 4.0 FOR AN S-C-S-S LAMINATE

is:

$$Y'''(y_o) = -A\,\{\alpha_m \cos(\alpha_m y_o) - \alpha_m \gamma_m \cos(\gamma_m y_o)\}$$
$$+ B\,\{\alpha_m \sin(\alpha_m y_o) - \gamma_m \sin(\gamma_m y_o)\} \qquad (258)$$

Substitution of equations (256), (257), and (258) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sin(\alpha_m b_o)\ \{-\alpha_m^2\} \\ + \sin(\gamma_m b_o)\ \{\alpha_m \gamma_m\} \end{array} & \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^2\} \\ + \cos(\gamma_m b_o)\ \{\gamma_m^2\} \end{array} \\ \hline \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^3 \\ \quad + K_{y_o}(\pi/a_o)^2 \alpha_m\} \\ + \cos(\gamma_m b_o)\ \{\alpha_m \gamma_m^2 \\ \quad - K_{y_o}(\pi/a_o)^2 \alpha_m\} \end{array} & \begin{array}{l} \sin(\alpha_m b_o)\ \{\alpha_m^3 \\ \quad - K_{y_o}(\pi/a_o)^2 \alpha_m\} \\ + \sin(\gamma_m b_o)\ \{-\gamma_m^3 \\ \quad + K_{y_o}(\pi/a_o)^2 \gamma_m\} \end{array} \end{vmatrix} = 0 \qquad (259)$$

Expansion of the determinant, division by the common multiple α_m, and multiplication by the quantity b_o^4 gives:

$$(\alpha_m b_o)(\gamma_m b_o) \sin(\alpha_m b_o) \sin(\gamma_m b_o)\ \{ (\gamma_m b_o)^2 + (\alpha_m b_o)^2$$
$$- 2K_{y_o}(\pi b_o/a_o)^2 \} + \cos(\alpha_m b_o) \cos(\gamma_m b_o)\ \{2(\alpha_m b_o)^2 (\gamma_m b_o)^2$$
$$- K_{y_o}(\pi b_o/a_o)^2 [(\alpha_m b_o)^2 + (\gamma_m b_o)^2]\} - (\alpha_m b_o)^4 - (\gamma_m b_o)^4$$
$$+ K_{y_o}(\pi b_o/a_o)^2 [(\alpha_m b_o)^2 + (\gamma_m b_o)^2] = 0 \qquad (260)$$

Attempts at solution of equation (260) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase are abandoned.

K_{y_o} Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, K_{y_o} can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that K_{y_o} range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that $Y(y_o)$ must take the form shown in equation (162). As usual, equations (228) through (231) make up the set of boundary conditions for the $Y(y_o)$ function. First, apply equation (228) to equation (162). This relation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (229), on the other hand, yields a relation between C_m and A_m such that $C_m = -(\alpha_m/\beta_m)A_m$ So equation (162) takes on the following form:

$$Y(y_o) = A_m\{\sin(\alpha_m y_o) - (\alpha_m/\beta_m)\sinh(\beta_m y_o)\} + B_m\{\cos(\alpha_m y_o) - \cosh(\beta_m y_o)\} \qquad (261)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{\alpha_m\cos(\alpha_m y_o) - \alpha_m\cosh(\beta_m y_o)\} - B_m\{\alpha_m\sin(\alpha_m y_o) + \beta_m\sinh(\beta_m y_o)\} \qquad (262)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = - A_m\{\alpha_m^2 \sin(\alpha_m y_o) + \alpha_m\beta_m \sinh(\beta_m y_o)\}$$
$$- B_m\{\alpha_m^2 \cos(\alpha_m y_o) + \beta_m^2 \cosh(\beta_m y_o)\} \qquad (263)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = - A_m\{\alpha_m^3 \cos(\alpha_m y_o) + \alpha_m\beta_m^2 \cosh(\beta_m y_o)\}$$
$$+ B_m\{\alpha_m^3 \sin(\alpha_m y_o) - \beta_m^3 \sinh(\beta_m y_o)\} \qquad (264)$$

Substitution of equations (262), (263), and (264) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o)\{\alpha_m^2\} + \sinh(\beta_m b_o)\{\alpha_m\beta_m\} & \cos(\alpha_m b_o)\{\alpha_m^2\} + \cosh(\beta_m b_o)\{\beta_m^2\} \\ \cos(\alpha_m b_o)\{-\alpha_m^3 + K_{y_o}(\pi/a_o)^2\alpha_m\} + \cosh(\beta_m b_o)\{-\alpha_m\beta_m^2 - K_{y_o}(\pi/a_o)^2\alpha_m\} & \sin(\alpha_m b_o)\{\alpha_m^3 - K_{y_o}(\pi/a_o)^2\alpha_m\} + \sinh(\beta_m b_o)\{-\beta_m^3 - K_{y_o}(\pi/a_o)^2\beta_m\} \end{vmatrix} = 0 \qquad (265)$$

Expansion of the determinant, division by the common multiple α_m, and multiplication by the quantity b_o^4 gives:

$$(\alpha_m b_o)(\beta_m b_o)\sin(\alpha_m b_o)\sinh(\beta_m b_o)\{(\alpha_m b_o)^2 - (\beta_m b_o)^2 - 2K_{y_o}(\pi b_o/a_o)^2\}$$
$$+ (\alpha_m b_o)^4 + (\beta_m b_o)^4 + K_{y_o}(\pi b_o/a_o)^2[(\beta_m b_o)^2 - (\alpha_m b_o)^2]$$
$$+ \cos(\alpha_m b_o)\cosh(\beta_m b_o)\{2(\alpha_m b_o)^2(\beta_m b_o)^2 + K_{y_o}(\pi b_o/a_o)^2[(\alpha_m b_o)^2 - (\beta_m b_o)^2]\} = 0 \qquad (266)$$

Equation (266) represents the governing equation for the combination of any plate aspect ratio and K_{y_o} less than or equal to zero. In other words, the roots of equation (266) yield the smallest values of K_{x_o} for any a_o/b_o and tensile or zero K_{y_o}. Consideration of results generated by equation (266) is postponed until the last of the three subcases is presented.

K_{y_o} Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, K_{y_o} can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Furthermore, equations (172), (173), and (174) sequentially illustrate that $Y(y_o)$ must take the form shown in equation (175). Note also that equations (228) through (231) again comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (228) to equation (175). This combination fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (229), on the other hand, yields a relation between C_m and A_m such that $C_m = -(\theta_m/\beta_m)A_m$ So equation (175) takes on the following form:

$$Y(y_o) = A_m\{ \sinh(\theta_m y_o) - (\theta_m/\beta_m) \sinh(\beta_m y_o) \}$$
$$+ B_m\{ \cosh(\theta_m y_o) - \cosh(\beta_m y_o) \} \qquad (267)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{\theta_m \cosh(\theta_m y_o) - \theta_m \cosh(\beta_m y_o) \}$$
$$+ B_m\{\theta_m \sinh(\theta_m y_o) - \beta_m \sinh(\beta_m y_o) \} \qquad (268)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m\{\theta_m^2 \sinh(\theta_m y_o) - \theta_m\beta_m \sinh(\beta_m y_o) \}$$
$$+ B_m\{\theta_m^2 \cosh(\theta_m y_o) - \beta_m^2 \cosh(\beta_m y_o) \} \qquad (269)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{\theta_m^3 \cosh(\theta_m y_o) - \theta_m \beta_m^2 \cosh(\beta_m y_o) \}$$
$$+ B_m\{\theta_m^3 \sinh(\theta_m y_o) - \beta_m^3 \sinh(\beta_m y_o) \} \qquad (270)$$

Substitution of equations (268), (269), and (270) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(\theta_m b_o)\ \{\theta_m^2\} + \sinh(\beta_m b_o)\ \{-\theta_m\beta_m\} & \cosh(\theta_m b_o)\ \{\theta_m^2\} + \cosh(\beta_m b_o)\ \{-\beta_m^2\} \\ \hline \cosh(\theta_m b_o)\ \{\theta_m^3 + K_{y_o}(\pi/a_o)^2\theta_m\} + \cosh(\beta_m b_o)\ \{-\theta_m\beta_m^2 - K_{y_o}(\pi/a_o)^2\theta_m\} & \sinh(\theta_m b_o)\ \{\theta_m^3 + K_{y_o}(\pi/a_o)^2\theta_m\} + \sinh(\beta_m b_o)\ \{-\beta_m^3 - K_{y_o}(\pi/a_o)^2\beta_m\} \end{vmatrix} = 0 \qquad (271)$$

Expansion of the determinant, division by the common multiple θ_m, and multiplication by the quantity $(-b_o^4)$ gives:

$$(\theta_m b_o)(\beta_m b_o)\sinh(\theta_m b_o)\sinh(\beta_m b_o)\{(\theta_m b_o)^2 + (\beta_m b_o)^2 + 2K_{y_o}(\Pi b_o/a_o)^2\} + (\theta_m b_o)^4 + (\beta_m b_o)^4 + K_{y_o}(\Pi b_o/a_o)^2[(\theta_m b_o)^2 + (\beta_m b_o)^2] - \cosh(\theta_m b_o)\cosh(\beta_m b_o)\{2(\theta_m b_o)^2(\beta_m b_o)^2 + K_{y_o}(\Pi b_o/a_o)^2[(\theta_m b_o)^2 + (\beta_m b_o)^2]\} = 0 \quad (272)$$

Attempts at solution of equation (272) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase as a whole are abandoned.

Discussion of Results

Table X gives selected a_o/b_o , K_{y_o} , and K_{x_o} ordered triplets as determined by equation (240). Note that the results generated by equation (240) are given--and indeed are only valid--for compressive or positive K_{y_o} . These numbers and plots based upon this data reveal two very important aspects of the buckling characteristics of a laminate supporterd by this relatively weak set of boundary conditions. First, all values of minimum K_{x_o} for any combination of a_o/b_o and compressive K_{y_o} are achieved when $m = 1$ is utilized in the terms which compose equation (240). More broadly, K_{x_o} plotted versus a_o/b_o for any constant compressive K_{y_o} results in just one continuous curve. There are no transition points to another curve.

Second, curves of K_{x_o} versus a_o/b_o for constant compressive K_{y_o} lie well below the zero ordinate for small and intermediate aspect ratios. However, these curves continuously trend upward as a_o/b_o increases. Indeed, as

a_o/b_o reaches a certain level, the proper K_{x_o} value lies just slightly and almost indistinguishably below the value characterizing the boundary curve $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ Since $m = 1$ in all cases for positive K_{y_o}, this boundary value of K_{x_o} is given by:

$$K_{x_o} = (b_o/a_o)^2\{1 - (K_{y_o}/2)\}^2 \qquad (273)$$

For example, virtually the last a_o/b_o and corresponding K_{x_o} coordinate which can be determined for $K_{y_o} = 3.00$ are $a_o/b_o = 4.011$ and $K_{x_o} = -0.0777436$ The boundary value of K_{x_o} for the noted y_o-buckling coefficient and plate aspect ratio is, by equation (273), -0.0776971 Figure 21 is a plot of just this asymptotic approach of K_{x_o} toward the boundary value of K_{x_o} given by equation (273) for constant $K_{y_o} = 3.00$

Note that as K_{y_o} increases, the value of a_o/b_o at which the K_{x_o} versus a_o/b_o plot becomes almost one with the curve defined by equation (273) is delayed. However, this merging is only postponed; it is never averted. As a result, as a_o/b_o becomes very large the limiting value of K_{x_o} for any positive K_{y_o} approaches zero (the limiting value of equation (273)).

Table XI gives selected a_o/b_o , K_{y_o} , and K_{x_o} ordered triplets as determined by equation (266). Note that the results generated by equation (266) are given and are only valid for negative or zero K_{y_o}. Also included is the integer value of m which produces this minimum K_{x_o} . Furthermore, each entry point which corresponds to a transition point from the m curve to the (m+1) curve is superscripted in the a_o/b_o

column with a star (*). Especially be aware that discontinuous curves as opposed to the continuous curves as discussed above depict K_{x_o} versus a_o/b_o plots for tensile K_{y_o} . The statistics presented in Table XI expose two key characteristics of laminates under tension in the y_o-direction.

First, the transition values of a_o/b_o increase as K_{y_o} becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative (or zero) magnitude of K_{y_o} , K_{x_o} attains a limiting value of 0.7125 as a_o/b_o approaches infinity. This is a different asymptotic value than that rendered by positive y_o -buckling coefficients.

Figure 22 represents a plot of K_{x_o} versus a_o/b_o for twelve distinct values of K_{y_o} . The lowest curve characterizes $K_{y_o} = 3.0$; whereas, the highest depicts $K_{y_o} = -5.0$ In ascending order the magnitudes of the y_o-buckling coefficients which correspond to these remaining curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0 This graph reinforces the concepts that K_{x_o} for a compressive K_{y_o} is determined by one continuous curve and that K_{x_o} for a tensile or zero K_{y_o} is rendered by the lowest ordinates of an infinite number of intersecting curves. In addition, the merging of the family of curves to two separate

asymptotes is readily apparent.

Figure 23 plots in three dimensions the same information as Figure 22. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 22; however, the quantitative aspect of Figure 23 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table XII gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o--1.0, 2.2, and 5.4 Figures 24, 25, and 26 represent two-dimensional plots at these constant a_o/b_o slices of 1.0, 2.2, and 5.4, respectively. Because the data for each of the curves are derived from two completely different equations ((240) for K_{y_o} positive; (266) for K_{y_o} zero or negative), a mild variance in slope in the vicinity of $K_{y_o} = 0.0$ is observed for $a_o/b_o = 1.0$ and increasingly larger variances for $a_o/b_o = 2.2$ and $a_o/b_o = 5.4$ These more distinct breaks typify the wide variance at relatively large aspect ratios between the x_o-buckling coefficient corresponding to negative K_{y_o} and the K_{x_o} due to positive K_{y_o}.

TABLE X

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Clamped and Free on the Two Edges Normal to the y_o-Direction (for K_{y_o} greater than zero)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	1.5	-3.4700
1	1.0000	1.5	-1.2983
1	1.4000	1.5	-0.8198
1	1.8000	1.5	-0.5835
1	2.2000	1.5	-0.3604
1	2.6000	1.5	-0.1102
1	1.0000	2.0	-3.0112
1	1.4000	2.0	-1.7026
1	1.8000	2.0	-1.1978
1	2.2000	2.0	-0.8624
1	2.6000	2.0	-0.5549
1	3.0000	2.0	-0.2351
1	1.0000	2.5	-5.2431
1	1.4000	2.5	-2.8129
1	1.8000	2.5	-1.9217
1	2.2000	2.5	-1.4272
1	2.6000	2.5	-1.0403
1	3.0000	2.5	-0.6750
1	3.4000	2.5	-0.2997

TABLE XI

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Clamped and Free on the Two Edges Normal to the y_o-Direction (for K_{y_o} less than or equal to zero)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	-5.0	4.2933
1	1.0000	-5.0	2.7468
1	1.4092*	-5.0	2.5178
2	2.0000	-5.0	1.6088
2	2.6000	-5.0	1.3175
2	3.2294*	-5.0	1.2465
3	3.8000	-5.0	1.0691
3	4.4000	-5.0	0.9843
3	5.0962*	-5.0	0.9626
1	0.6000	-2.0	3.4638
1	1.2000	-2.0	1.6382
1	1.7951*	-2.0	1.5516
2	2.4000	-2.0	1.0951
2	3.0000	-2.0	0.9534
2	3.6930*	-2.0	0.9532
3	4.2000	-2.0	0.8607
3	4.8000	-2.0	0.8179
3	5.4000	-2.0	0.8227
1	0.6000	0.0	2.8235
1	1.2000	0.0	0.8772
1	1.8000	0.0	0.7198
1	2.3694*	0.0	0.8906
2	3.0000	0.0	0.7300
2	3.6000	0.0	0.7198
2	4.1039*	0.0	0.7719
3	4.8000	0.0	0.7155
3	5.4000	0.0	0.7198

TABLE XII

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported in the x_o-Direction and Clamped and Free on the Two Edges Normal to the y_o-Direction

m	a_o/b_o	K_{y_o}	K_{x_o}
1	1.0000	-5.0	2.7468
1	1.0000	-3.0	2.1608
1	1.0000	-1.0	1.5228
1	1.0000	0.5	0.6820
1	1.0000	1.0	-0.0888
1	1.0000	1.5	-1.2983
1	1.0000	2.0	-3.0112
1	1.0000	2.5	-5.2431
2	2.2000	-5.0	1.4730
2	2.2000	-3.0	1.2922
2	2.2000	-1.0	1.0921
1	2.2000	1.0	0.0841
1	2.2000	1.5	-0.3604
1	2.2000	2.0	-0.8624
1	2.2000	2.5	-1.4272
1	2.2000	3.0	-2.0602
4	5.4000	-5.0	0.9212
4	5.4000	-3.0	0.8661
3	5.4000	-1.0	0.7717
1	5.4000	0.5	0.0321
1	5.4000	1.0	0.0257
1	5.4000	2.0	0.0000
1	5.4000	2.5	-0.0193
1	5.4000	3.0	-0.0429

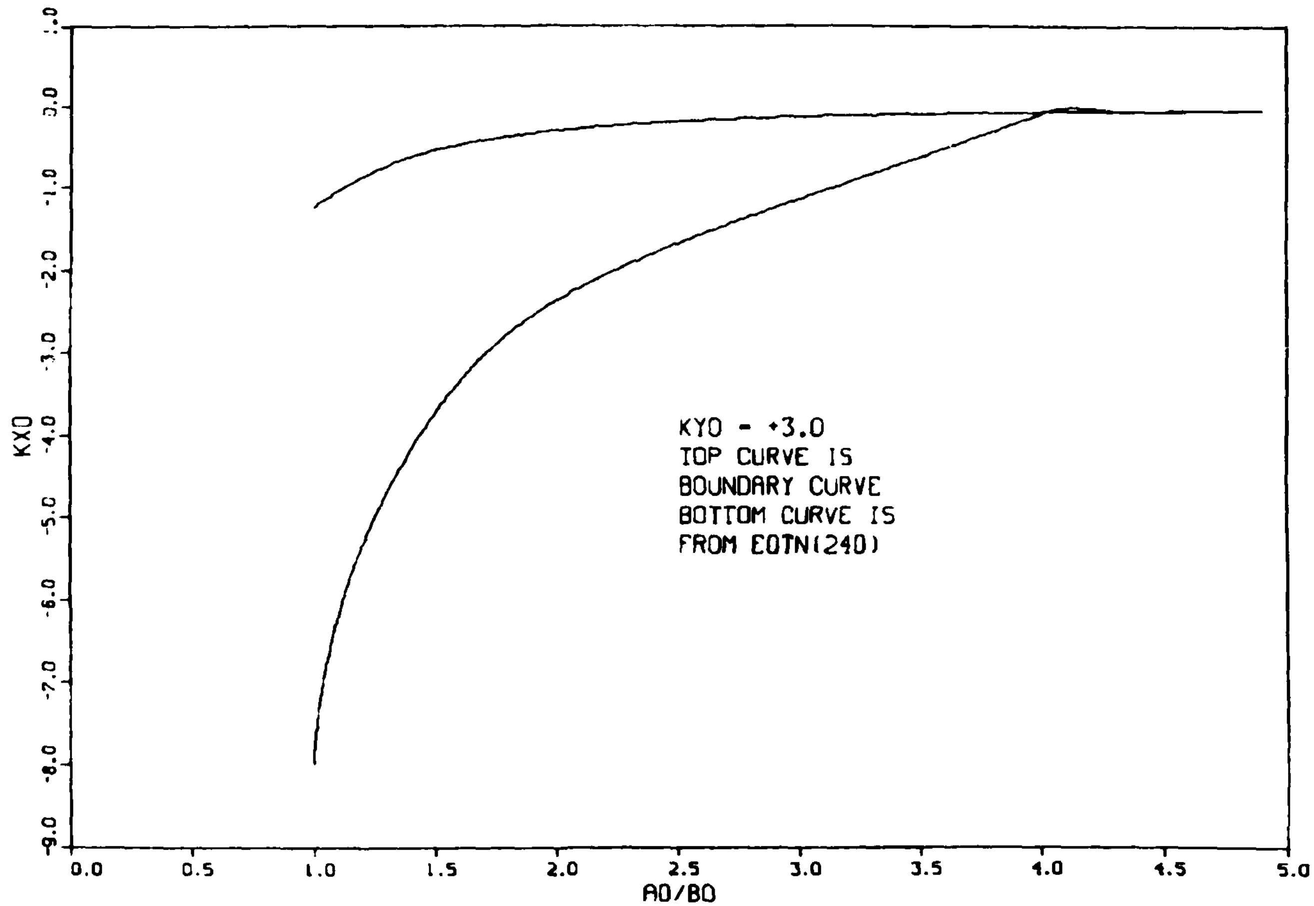


FIGURE 21

APPROACH OF XO-BUCKLING COEFFICIENT TO BOUNDARY CURVE FOR AN S-C-S-F LAMINATE

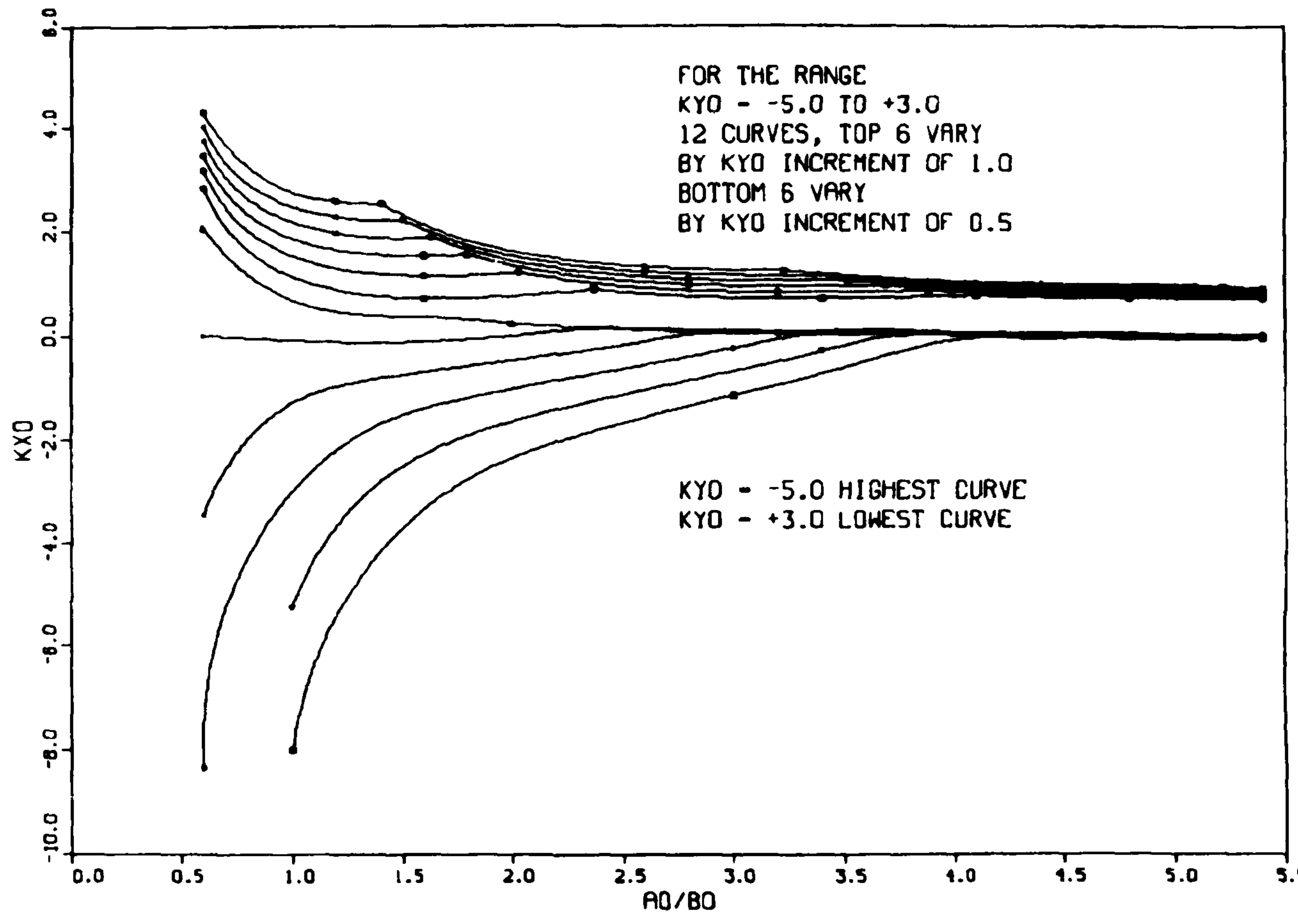


FIGURE 22

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-C-S-F LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

Expansion of the determinant and multiplication by the quantity b_o^4 results in the following system of equations (the second equation a simplification of the first):

$$\sin^2(Ub_o)\ \{-(Ub_o)^5 + K_{y_o}(\pi b_o/a_o)^2 (Ub_o)^3\}$$
$$+ \sin(Ub_o)\cos(Ub_o)\ \{(Ub_o)^4 + K_{y_o}(\pi b_o/a_o)^2 (Ub_o)^2\}$$
$$+ \cos(Ub_o)\ \{-(Ub_o)^5 + K_{y_o}(\pi b_o/a_o)^2 (Ub_o)^3\} = 0 \qquad (304)$$

$$\sin(Ub_o)\cos(Ub_o)\ \{(Ub_o)^4 + K_{y_o}(\pi b_o/a_o)^2 (Ub_o)^2\}$$
$$- (Ub_o)^5 + K_{y_o}(\pi b_o/a_o)^2 (Ub_o)^3 = 0 \qquad (305)$$

No value of (Ub_o) greater than zero can satisfy equation (305). Therefore, no possible solutions exist for the present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} > 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $Y(y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of K_{y_o} for valid solutions. Three subcases must be considered so that a solution for K_{x_o} may be determined for any range of K_{y_o}.

function. The enforcement of equation (277) on equation (132) dictates that B_m must vanish. In addition, application of equation (278) leads one to the conclusion that C_m is zero. So equation (132) takes on the following form:

$$Y(y_o) = A_m \sin(Uy_o) + D_m y_o \cos(Uy_o) \tag{299}$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m U \cos(Uy_o) + D_m\{\cos(Uy_o) - Uy_o \sin(Uy_o)\} \tag{300}$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m U^2 \sin(Uy_o) + D_m\{-2U \sin(Uy_o) - U^2 y_o \cos(Uy_o)\} \tag{301}$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = -A_m U^3 \cos(Uy_o) + D_m\{-3U^2 \cos(Uy_o) + U^3 y_o \sin(Uy_o)\} \tag{302}$$

Substitution of equations (300), (301), and (302) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and D_m. For non-trivial A_m and D_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(Ub_o)\{-U^2\} & \sin(Ub_o)\{-2U\} + \cos(Ub_o)\{-U^2 b_o\} \\ \cos(Ub_o)\{-U^3 + K_{y_o}(\pi/a_o)^2 U\} & \sin(Ub_o)\{U^3 b_o - K_{y_o}(\pi/a_o)^2 U b_o\} + \cos(Ub_o)\{-3U^2 + K_{y_o}(\pi/a_o)^2\} \end{vmatrix} = 0 \tag{303}$$

$K_{y_o} = 0.$

For $K_{y_o} = 0$ equation (128) constitutes the required shape of the unknown $Y(y_o)$ function. In addition, equations (277) through (280) are the sets of constraints for this $Y(y_o)$ function. Moreover, notice that equation (280) reduces to $Y'''(b_o) = 0$ in this instance since $K_{y_o} = 0$ Equations (277) and (278) imply that $A_m = C_m = 0$ Equation (280) similarly necessitates that D_m vanish. After the imposition of these three conditions, equation (128) reduces to:

$$Y(y_o) = B_m y_o \tag{297}$$

Equation (279) is identically satisfied for this $Y(y_o)$ given by equation (297). All boundary conditions, equations (277) through (280), are therefore upheld by this $Y(y_o)$ expressed in equation (297). As a result, if $K_{y_o} = 0$, $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ is a valid solution. Rearrangement of this relation gives:

$$K_{x_o} = [\, m \,/\, (a_o/b_o)\,]^2 \tag{298}$$

When m is set equal to unity, equation (298) constitutes the minimum K_{x_o} for any plate aspect ratio and $K_{y_o} = 0$ Consideration of results generated by equation (298) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

$K_{y_o} > 0.$

For $K_{y_o} > 0$ equations (130) and (131) show that the unknown function $Y(y_o)$ must fit the relation given by equation (132). Equations (277) through (280) once more comprise the group of boundary conditions for the $Y(y_o)$

equations (279) and (280), along with basic algebraic manipulation, yield two homogeneous linear equations in coefficients A_m and D_m. For non-trivial A_m and D_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(Tb_o)\ \{T^2\} & \sinh(Tb_o)\ \{2T\} + \cosh(Tb_o)\ \{T^2 b_o\} \\ \cosh(Tb_o)\ \{T^3 + K_{y_o}(\pi/a_o)^2 T\} & \sinh(Tb_o)\ \{T^3 b_o + K_{y_o}(\pi/a_o)^2 Tb_o\} + \cosh(Tb_o)\ \{3T^2 + K_{y_o}(\pi/a_o)^2\} \end{vmatrix} = 0 \tag{294}$$

Expansion of the determinant and multiplication by the quantity b_o^4 results in the following equations (the second equation a simplification of the first):

$$\begin{aligned} &\sinh^2(Tb_o)\ \{ (Tb_o)^5 + K_{y_o}(\pi b_o/a_o)^2 (Tb_o)^3 \} \\ &+ \sinh(Tb_o)\cosh(Tb_o)\ \{ (Tb_o)^4 - K_{y_o}(\pi b_o/a_o)^2 (Tb_o)^2 \} \\ &- \cosh^2(Tb_o)\ \{ (Tb_o)^5 + K_{y_o}(\pi b_o/a_o)^2 (Tb_o)^3 \} = 0 \end{aligned} \tag{295}$$

$$\begin{aligned} &\sinh(Tb_o)\cosh(Tb_o)\ \{ (Tb_o)^4 - K_{y_o}(\pi b_o/a_o)^2 (Tb_o)^2 \} \\ &\qquad - (Tb_o)^5 - K_{y_o}(\pi b_o/a_o)^2 (Tb_o)^3 = 0 \end{aligned} \tag{296}$$

No value of Tb_o greater than zero can satisfy equation (296). Therefore, no possible solutions exist for the present boundary conditions for

$\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} = 0$ and $K_{y_o} < 0$

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 -1\} = 0$ equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of K_{y_o}, each possible range of K_{y_o} --negative, zero, and positive--will be analyzed as different subcases.

$K_{y_o} < 0$.

For $K_{y_o} < 0$ equations (121) and (122) lead to the conclusion that the unknown function $Y(y_o)$ must take the form shown in equation (123). Equations (277) through (280) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (277) on equation (123) dictates that B_m must vanish. In addition, application of equation (278) leads one to the conclusion that C_m is zero. So equation (123) takes on the following form:

$$Y(y_o) = A_m\sinh(Ty_o) + D_m y_o \cosh(Ty_o) \qquad (290)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m T\cosh(Ty_o) + D_m\{\cosh(Ty_o) + Ty_o\sinh(Ty_o)\} \qquad (291)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m T^2\sinh(Ty_o) + D_m\{ 2T\sinh(Ty_o) + T^2 y_o\cosh(Ty_o)\} \qquad (292)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m T^3\cosh(Ty_o) + D_m\{ 3T^2\cosh(Ty_o) + T^3 y_o\sinh(Ty_o)\} \qquad (293)$$

Substitution of equations (291), (292), and (293) into

Expansion of the determinant and multiplication by the quantity b_o^5 results in the following equations (the second equation a simplification of the first):

$$\sin^2(sb_o)\ [\ e^{2cb_o}\{-(cb_o)^4(sb_o) - 2\ (cb_o)^2(sb_o)^3 - (sb_o)^5$$
$$+ K_{y_o}(\pi b_o/a_o)^2\ (\ (cb_o)^2(sb_o) + (sb_o)^3\ )\}$$
$$+ e^{-2cb_o}\{\ (cb_o)^4(sb_o) + 2\ (cb_o)^2(sb_o)^3 + (sb_o)^5$$
$$- K_{y_o}(\pi b_o/a_o)^2\ (\ (cb_o)^2(sb_o) + (sb_o)^3\ )\}]$$
$$+\cos^2(sb_o)\ [\ e^{2cb_o}\{-(cb_o)^4(sb_o) - 2\ (cb_o)^2(sb_o)^3 - (sb_o)^5$$
$$+ K_{y_o}(\pi b_o/a_o)^2\ (\ (cb_o)^2(sb_o) + (sb_o)^3\ )\}$$
$$+ e^{-2cb_o}\{\ (cb_o)^4(sb_o) + 2\ (cb_o)^2(sb_o)^3 + (sb_o)^5$$
$$- K_{y_o}(\pi b_o/a_o)^2\ (\ (cb_o)^2(sb_o) + (sb_o)^3\ )\}]$$
$$+ \sin(sb_o)\cos(sb_o)\ [4\ (cb_o)^5 + 4(cb_o)(sb_o)^4 + 8(cb_o)^3(sb_o)^2$$
$$+ K_{y_o}(\pi b_o/a_o)^2\{\ 4\ (cb_o)^3 + 4\ (cb_o)\ (sb_o)^2\}] = 0 \qquad (288)$$

$$(e^{2cb_o} - e^{-2cb_o})\ \{-(cb_o)^4(sb_o) - 2\ (cb_o)^2(sb_o)^3 - (sb_o)^5$$
$$+ K_{y_o}(\pi b_o/a_o)^2[\ (cb_o)^2(sb_o) + (sb_o)^3\ ]\ \}$$
$$+ 4\sin(sb_o)\cos(sb_o)\ \{\ (cb_o)^5 + (cb_o)\ (sb_o)^4$$
$$+ 2\ (cb_o)^3(sb_o)^2 + K_{y_o}(\pi b_o/a_o)^2[\ (cb_o)^3$$
$$+ (cb_o)\ (sb_o)^2\ ]\ \} = 0 \qquad (289)$$

Equation (289) constitutes the governing equation for any plate aspect ratio and K_{y_o} greater than zero. In other words, the roots of equation (289) yield the smallest values of K_{x_o} for any a_o/b_o and compressive K_{y_o}. Consideration of results generated by equation (289) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

$$Y''(y_o) = A_m\{ (c^2 - s^2) \sin(sy_o) [e^{cy_o} + e^{-cy_o}]$$
$$+ (2cs) \cos(sy_o) [e^{cy_o} - e^{-cy_o}] \}$$
$$+ B_m\{ (c^2 - s^2) \cos(sy_o) [e^{cy_o} - e^{-cy_o}]$$
$$- (2cs) \sin(sy_o) [e^{cy_o} + e^{-cy_o}] \} \qquad (285)$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{(3c^2 s - s^3)\cos(sy_o) [e^{cy_o} + e^{-cy_o}]$$
$$+ (c^3 - 3cs^2)\sin(sy_o) [e^{cy_o} - e^{-cy_o}]\}$$
$$+ B_m\{ (s^3 - 3c^2 s)\sin(sy_o) [e^{cy_o} - e^{-cy_o}]$$
$$+ (c^3 - 3cs^2)\cos(sy_o) [e^{cy_o} + e^{-cy_o}]\} \qquad (286)$$

Substitution of equations (284), (285), and (286) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sin(sb_o)[(c^2 - s^2)(e^{cb_o} + e^{-cb_o})] \\ + \cos(sb_o)[(2cs)(e^{cb_o} - e^{-cb_o})] \end{array} & \begin{array}{l} \sin(sb_o)[(-2cs)(e^{cb_o} + e^{-cb_o})] \\ + \cos(sb_o)[(c^2 - s^2)(e^{cb_o} - e^{-cb_o})] \end{array} \\ \begin{array}{l} \sin(sb_o)[(c^3 - 3cs^2 + K_{y_o} (\Pi/a_o)^2 c)\{e^{cb_o} - e^{-cb_o} \}] \\ + \cos(sb_o)[(3c^2 s - s^3 + K_{y_o} (\Pi/a_o)^2 s)\{e^{cb_o} + e^{-cb_o} \}] \end{array} & \begin{array}{l} \sin(sb_o)[(s^3 - 3c^2 s - K_{y_o} (\Pi/a_o)^2 s)\{e^{cb_o} - e^{-cb_o} \}] \\ + \cos(sb_o)[(c^3 - 3cs^2 + K_{y_o} (\Pi/a_o)^2 c)\{e^{cb_o} + e^{-cb_o} \}] \end{array} \end{vmatrix} = 0 \qquad (287)$$

For equations (275) and (276) to have meaning in the general case, the following conditions must hold:

$$Y(0) = 0 \qquad (277)$$

$$Y''(0) = 0 \qquad (278)$$

$$Y''(b_o) = 0 \qquad (279)$$

$$Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o) = 0 \qquad (280)$$

First, apply equation (277) to equation (106). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (278) on the $Y(y_o)$ equation similarly determines a value C_m in terms of A_m.

$$A_m(2cs) + B_m(c^2 - s^2) - C_m(2cs) + D_m(c^2 - s^2) = 0 \qquad (281)$$

But since $D_m = -B_m$,

$$C_m = A_m(2cs)/(2cs) = A_m \qquad (282)$$

For these values of C_m and D_m, equation (106) takes on the following form:

$$Y(y_o) = A_m\{ \sin(sy_o)\ (e^{cy_o} + e^{-cy_o}) \}$$
$$+ B_m\{ \cos(sy_o)\ (e^{cy_o} - e^{-cy_o}) \} \qquad (283)$$

The first derivative of $Y(y_o)$ with respect to y_o is therefore:

$$Y'(y_o) = A_m\{ \cos(sy_o)\ (se^{cy_o} + se^{-cy_o})$$
$$+ \sin(sy_o)\ (ce^{cy_o} - ce^{-cy_o}) \}$$
$$+ B_m\{ \sin(sy_o)\ (-se^{cy_o} + se^{-cy_o})$$
$$+ \cos(sy_o)\ (ce^{cy_o} + ce^{-cy_o}) \} \qquad (284)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

a lack of symmetry is present in the boundary conditions in the y_o-direction.

Just as before a displacement function w which satisfies the first two stipulations of equation (274) is given by equation (77). Furthermore, substitution of this relation for w into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable r is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states--negative, zero, and positive--of the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$ equations (82) through (105) illustrate that the unknown function $Y(y_o)$ must take the form shown in equation (106).

Consider now the boundary conditions, equations (274), for this case of a laminate simply supported on three sides and free on the fourth. When the chosen form of w, equation (77), is substituted into the final three lines of equations (274), the following must hold:

$$Y(0)\sin(m\pi x_o/a_o) = 0 \quad ; \quad Y''(0)\sin(m\pi x_o/a_o) = 0 \qquad (275)$$

$$Y''(b_o)\sin(m\pi x_o/a_o) = 0 \quad ;$$

$$[Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o)]\sin(m\pi x_o/a_o) = 0 \qquad (276)$$

VII. Flat Rectangular Composite Laminate Simply Supported in the x_o-Direction and Simply Supported and Free on the Two Edges Normal to the y_o-Direction

The boundary conditions for a laminate simply supported in the x_o-direction and simply supported and free on the two edges perpendicular to the y_o-direction display no symmetry in the y_o-direction. For the two edges which have normals parallel to the x_o-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. Similarly, for that edge normal to the y_o-direction which is simply supported, these same edge conditions hold. However, for the remaining edge which is oriented perpendicular to the y_o-direction, the vanishing of the normal component of the moment to this free edge and the satisfaction of equation (224) constitute the two requisites. In equation form, the following must hold:

$$\begin{aligned}
&\text{on edge } x_o=-a_o/2\ , && w = 0\ ;\ w_{,x_o x_o} = 0 \\
&\text{on edge } x_o= a_o/2\ , && w = 0\ ;\ w_{,x_o x_o} = 0 \\
&\text{on edge } y_o= 0\ , && w = 0\ ;\ w_{,y_o y_o} = 0 \qquad\qquad (274)\\
&\text{on edge } y_o= b_o\ , && w_{,y_o y_o} = 0\ ; \\
& && w_{,y_o y_o y_o} + K_{y_o}(\pi/a_o)^2 w_{,y_o} = 0
\end{aligned}$$

Note that the origin of coordinates in the affine space is taken to be at the center of the simply supported edge normal to the y_o-direction. This choice of origin location, in general, allows maximum simplicity in manipulations since

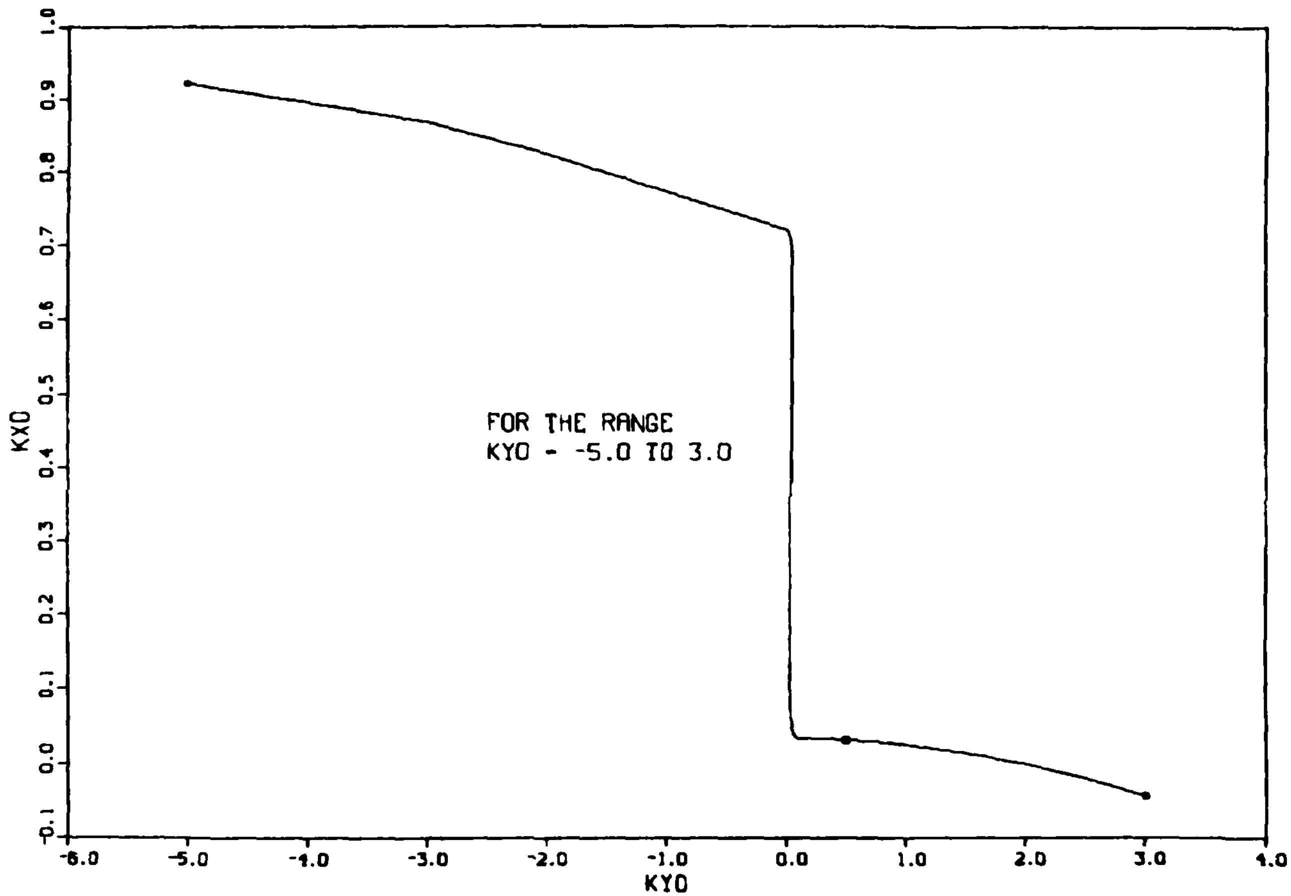


FIGURE 26

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 5.4 FOR AN S-C-S-F LAMINATE

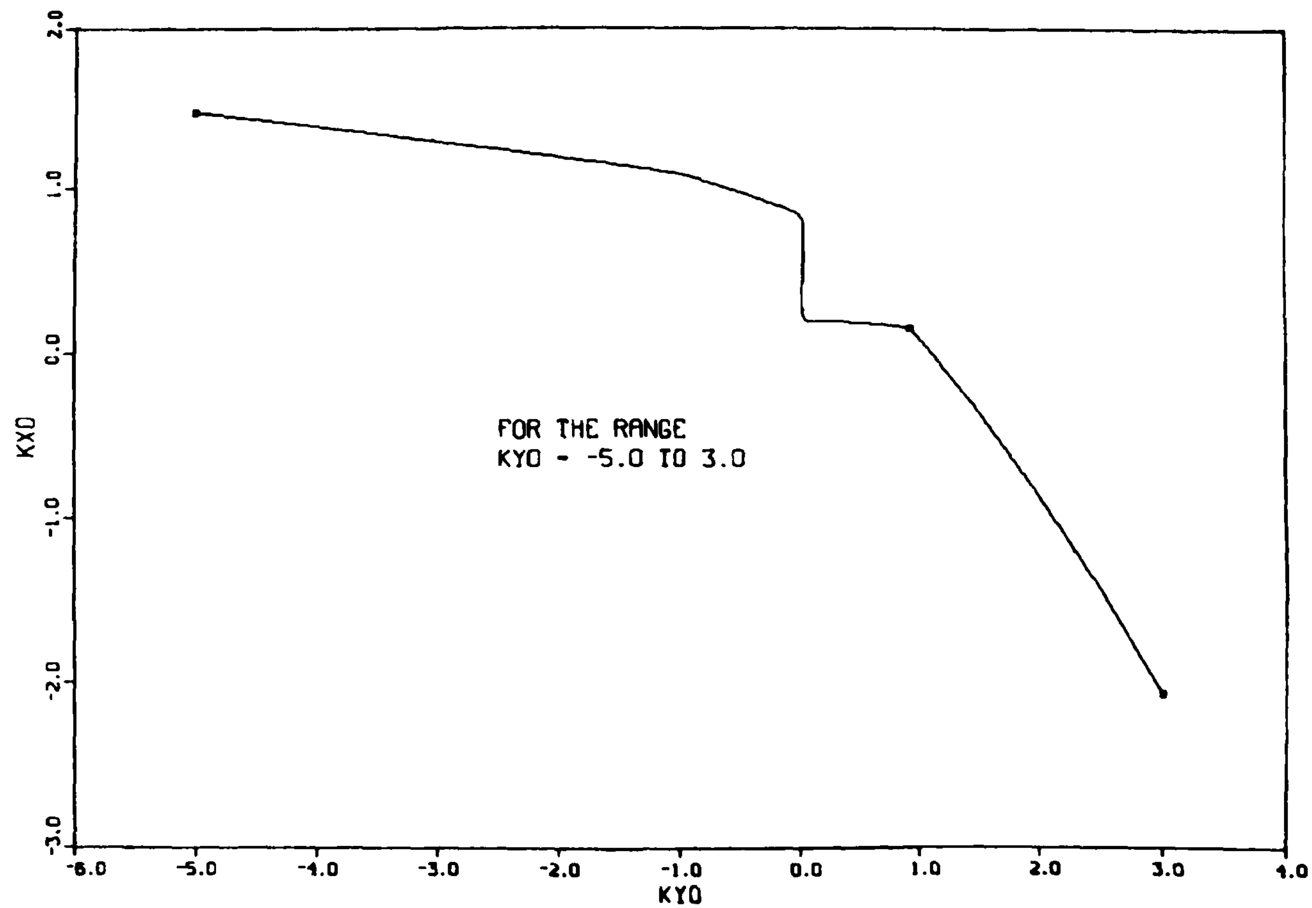


FIGURE 25

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.2 FOR AN S-C-S-F LAMINATE

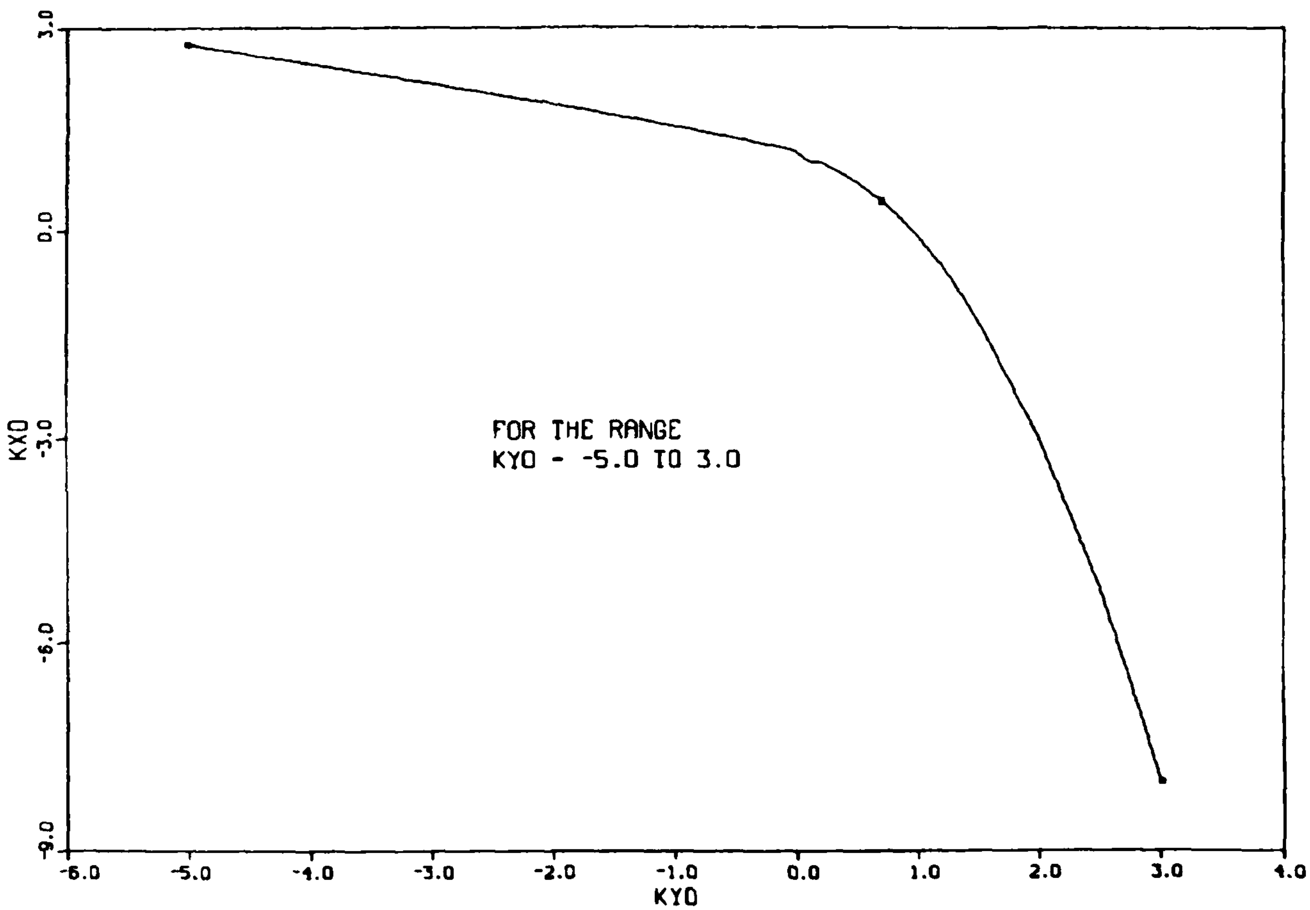


FIGURE 24

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.0 FOR AN S-C-S-F LAMINATE

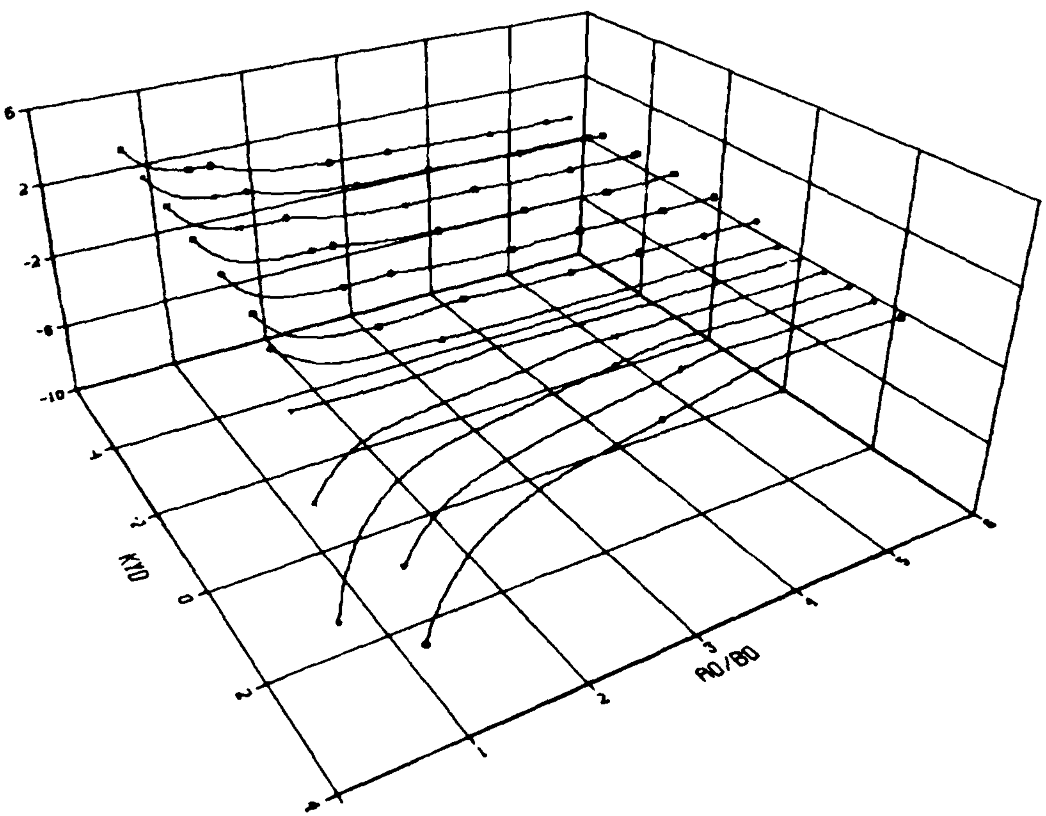


FIGURE 23

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO FOR AN S-C-S-F LAMINATE

K_{y_o} Ranges from a Relatively Large Positive Number to Positive Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, K_{y_o} can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (277) through (280) once more comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (277) to equation (144). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (278) on the $Y(y_o)$ equation, on the other hand, forces B_m and hence D_m to vanish. So equation (144) takes on the following form:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + C_m \sin(\nu_m y_o) \qquad (306)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m \alpha_m \cos(\alpha_m y_o) + C_m \nu_m \cos(\nu_m y_o) \qquad (307)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m \alpha_m^2 \sin(\alpha_m y_o) - C_m \nu_m^2 \sin(\nu_m y_o) \qquad (308)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = -A_m\alpha_m^3\cos(\alpha_m y_o) - C_m\gamma_m^3\cos(\gamma_m y_o) \qquad (309)$$

Substitution of equations (307), (308), and (309) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o)\ \{-\alpha_m^2\} & \sin(\gamma_m b_o)\ \{-\gamma_m^2\} \\ \cos(\alpha_m b_o)\ \{-\alpha_m^3 + K_{y_o}(\pi/a_o)^2\,\alpha_m\} & \cos(\gamma_m b_o)\ \{-\gamma_m^3 + K_{y_o}(\pi/a_o)^2\,\gamma_m\} \end{vmatrix} = 0 \qquad (310)$$

Expansion of the determinant, division by the common multiple $(\alpha_m\gamma_m)$, and multiplication by the quantity b_o^3 gives:

$$\sin(\alpha_m b_o)\cos(\gamma_m b_o)\ \{(\alpha_m b_o)(\gamma_m b_o)^2 - K_{y_o}(\pi b_o/a_o)^2(\alpha_m b_o)\}$$
$$+ \cos(\alpha_m b_o)\sin(\gamma_m b_o)\ \{(-(\alpha_m b_o)^2(\gamma_m b_o)$$
$$+ K_{y_o}(\pi b_o/a_o)\ (\gamma_m b_o)\} = 0 \qquad (311)$$

Attempts at solution of equation (311) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase are abandoned.

K_{y_o} <u>Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative

number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, K_{y_o} can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that K_{y_o} range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that $Y(y_o)$ must take the form shown in equation (162). As usual, equations (277) through (280) make up the set of boundary conditions for the $Y(y_o)$ function. First, apply equation (277) to equation (162). This stipulation fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (278) on the $Y(y_o)$ equation, on the other hand, forces B_m and hence D_m to vanish. So equation (162) takes on the following form:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + C_m \sinh(\beta_m y_o) \qquad (312)$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m \alpha_m \cos(\alpha_m y_o) + C_m \beta_m \cosh(\beta_m y_o) \qquad (313)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m \alpha_m^2 \sin(\alpha_m y_o) + C_m \beta_m^2 \sinh(\beta_m y_o) \qquad (314)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = -A_m \alpha_m^3 \cos(\alpha_m y_o) + C_m \beta_m^3 \cosh(\beta_m y_o) \qquad (315)$$

Substitution of equations (313), (314), and (315) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in

coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sin(\alpha_m b_o)\ \{-\alpha_m^2\} & \sinh(\beta_m b_o)\ \{\beta_m^2\} \\ \hline \cos(\alpha_m b_o)\ \{-\alpha_m^3 + K_{y_o}(\Pi/a_o)^2\alpha_m\} & \cosh(\beta_m b_o)\ \{\beta_m^3 + K_{y_o}(\Pi/a_o)^2\beta_m\} \end{vmatrix} = 0 \qquad (316)$$

Expansion of the determinant, division by the common multiple $(\alpha_m\beta_m)$, and multiplication by the quantity b_o^3 gives:

$$\sin(\alpha_m b_o)\cosh(\beta_m b_o)\{-(\alpha_m b_o)(\beta_m b_o)^2 - K_{y_o}(\Pi b_o/a_o)^2(\alpha_m b_o)\}$$
$$+ \cos(\alpha_m b_o)\sinh(\beta_m b_o)\{(\alpha_m b_o)^2(\beta_m b_o) - K_{y_o}(\Pi b_o/a_o)^2(\beta_m b_o)\} = 0 \qquad (317)$$

Equation (317) represents the governing equation for the combination of any plate aspect ratio and K_{y_o} less than zero. In other words, the roots of equation (317) yield the smallest values of K_{x_o} for any a_o/b_o and tensile K_{y_o}. Consideration of results generated by equation (317) is postponed until the last of the three subcases is presented.

K_{y_o} Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, K_{y_o} can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited

by the two inequalities expressed in equations (170) and (171). Furthermore, equations (172), (173), and (174) sequentially illustrate that $Y(y_o)$ must take the form shown in equation (175). Note also that equations (277) through (280) again comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (277) to equation (175). This combination fixes D_m in terms of B_m such that $D_m = -B_m$ Utilization of equation (278) on the $Y(y_o)$ equation, in contrast, forces B_m and hence D_m to vanish. So equation (175) takes on the following form:

$$Y(y_o) = A_m \sinh(\theta_m y_o) + C_m \sinh(\beta_m y_o) \tag{318}$$

The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m \theta_m \cosh(\theta\ y_o) + C_m \beta_m \cosh(\beta_m y_o) \tag{319}$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m \theta_m^2 \sinh(\theta_m y_o) + C_m \beta_m^2 \sinh(\beta_m y_o) \tag{320}$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m \theta_m^3 \cosh(\theta_m y_o) + C_m \beta_m^3 \cosh(\beta_m y_o) \tag{321}$$

Substitution of equations (319), (320), and (321) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

$$\begin{vmatrix} \sinh(\theta_m b_o)\ \{\theta_m^2\} & \sinh(\beta_m b_o)\ \{\beta_m^2\} \\ \cosh(\theta_m b_o)\ \{\theta_m^3 + K_{y_o}(\pi/a_o)^2\,\theta_m\} & \cosh(\beta_m b_o)\ \{\beta_m^3 + K_{y_o}(\pi/a_o)^2\,\beta_m\} \end{vmatrix} = 0 \qquad (322)$$

Expansion of the determinant, division by the common multiple $(\theta_m\beta_m)$, and multiplication by the quantity b_o^3 gives:

$$\begin{aligned} &\sinh(\theta_m b_o)\ \cosh(\beta_m b_o)\ \{\ (\theta_m b_o)(\beta_m b_o)^2 \\ &\qquad\qquad + K_{y_o}(\pi b_o/a_o)^2\,(\theta_m b_o)\} \\ &- \sinh(\beta_m b_o)\ \cosh(\theta_m b_o)\ \{\ (\theta_m b_o)^2\,(\beta_m b_o) \\ &\qquad\qquad + K_{y_o}(\pi b_o/a_o)^2\,(\beta_m b_o)\} = 0 \end{aligned} \qquad (323)$$

Attempts at solution of equation (323) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase as a whole are abandoned.

Discussion of Results

Table XIII gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets as determined by equation (289). Note that the results generated by equation (289) are given--and indeed are only valid--for compressive or positive K_{y_o}. These numbers and plots based upon this data reveal two very important aspects of the buckling characteristics of a laminate supported by this relatively weak set of boundary conditions. First, all values of minimum K_{x_o} for any combination of a_o/b_o and compressive K_{y_o} are achieved when $m = 1$ is utilized in the terms which compose equation (289). More broadly, K_{x_o} plotted versus a_o/b_o for any constant compressive K_{y_o}

results in just one continuous curve. There are no transition points to another curve.

Second, curves of K_{x_o} versus a_o/b_o for constant compressive K_{y_o} lie well below the zero ordinate for small and intermediate affine aspect ratios. However, these curves continuously trend upward as a_o/b_o increases. For any constant K_{y_o}, the K_{x_o} versus a_o/b_o plot asymptotically approaches a value of roughly $-0.3\ K_{y_o}$. For example, the following data reflects x_o-buckling coefficients (and corresponding y_o-buckling coefficients) for an affine aspect ratio of 60.0 :

a_o/b_o	K_{y_o}	K_{x_o}
60.0	0.5	-0.1517
60.0	1.0	-0.3038
60.0	1.5	-0.4560
60.0	2.0	-0.6082
60.0	2.5	-0.7604
60.0	3.0	-0.9128

Table XIV gives selected a_o/b_o, K_{x_o} ordered pairs for zero K_{y_o} only. This data is based exclusively upon equation (298). These numbers again bring to light two key details of the uniaxial ($K_{y_o} = 0$) buckling phenomenon. First, identical to the descriptions presented above, all values of minimum K_{x_o} for any affine aspect ratio and null K_{y_o} occur when $m = 1$ is inserted into equation (298). As a result, a

sketch of K_{x_o} versus a_o/b_o for $K_{y_o} = 0$ plots as one continuous curve; transition points are absent.

Second, the K_{x_o} versus a_o/b_o curve for $K_{y_o} = 0$ lies above the zero ordinate for all aspect ratios. As a_o/b_o becomes large, however, the $K_{x_o} = 0$ value is rapidly approached. Indeed, the limiting value of K_{x_o} for this uniaxial buckling case is in fact zero (as can be easily seen by examination of equation (298)).

Table XV gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered triplets as determined by equation (317). Note that the results generated by equation (317) are given and are only valid for negative K_{y_o}. Also included is the integer value of m which produces this minimum K_{x_o}. Furthermore, each entry point which corresponds to a transition point from the m curve to the (m + 1) curve is superscripted with a star (*). Especially be aware that discontinuous curves, as opposed to the continuous curves discussed above, depict K_{x_o} versus a_o/b_o plots for tensile K_{y_o}. The statistics presented in Table XV expose two key characteristics of laminates under tension in the y_o-direction. First, the transition values of a_o/b_o increase as K_{y_o} becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative magnitude of K_{y_o}, K_{x_o} attains a limiting value of zero as a_o/b_o approaches infinity. For this set of boundary conditions the rate of

approach toward this zero asymptote is relatively slow. For example, for $K_{y_o} = -5.0$, $K_{x_o} = 0.0493$ at $a_o/b_o = 50.0$ and $K_{x_o} = 0.0247$ at $a_o/b_o = 100.0$ Note that this asymptotic value of zero differs from those asymptotes presented for compressive K_{y_o}.

Figure 27 represents a plot of K_{x_o} versus a_o/b_o for twelve distinct values of K_{y_o} . The lowest curve characterizes $K_{y_o} = 3.0$; whereas, the highest depicts $K_{y_o} = -5.0$ In ascending order the magnitudes of the y_o-buckling coefficients which correspond to the remaining ten curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0 This graph reinforces the concepts that K_{x_o} for a compressive or zero K_{y_o} is determined by one continuous curve and that K_{x_o} for a tensile K_{y_o} is rendered by the lowest values of an infinite number of intersecting curves. In addition, the merging of the family of curves to the zero asymptote for tensile or zero K_{y_o} and to distinct, relatively evenly spaced asymptotes for compressive K_{y_o} is readily apparent.

Figure 28 plots in three dimensions the same information as Figure 27. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 27; however, the quantitative aspect of Figure 28 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table XVI gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o--1.4, 3.0, and 5.4 Figures 29, 30, and 31 represent two-dimensional plots at these constant a_o/b_o slices of 1.4, 3.0, and 5.4, respectively. Because the data for each of the curves are derived from three completely different equations ((289) for K_{y_o} positive; (298) for K_{y_o} zero; (317) for K_{y_o} negative), a mild variance of slope in the vicinity of $K_{y_o} = 0.0$ is observed for $a_o/b_o = 1.4$ and slightly larger variances for $a_o/b_o = 3.0$ and $a_o/b_o = 5.4$ These breaks are decidedly less severe than those observed in corresponding plots for a laminate simply supported on opposite sides and clamped and free on the remaining edges.

TABLE XIII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Simply Supported and Free on the Two Edges Normal to the y_o-Direction (for K_{y_o} less than zero)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	1.5	-3.4724
1	0.8000	1.5	-1.9413
1	1.2000	1.5	-0.8630
1	1.8000	1.5	-0.5403
1	2.4000	1.5	-0.4839
1	3.2000	1.5	-0.4658
1	4.0000	1.5	-0.4606
1	4.8000	1.5	-0.4586
1	5.4000	1.5	-0.4578
1	0.8000	2.0	-4.6837
1	1.0000	2.0	-2.9756
1	1.2000	2.0	-2.0506
1	1.6000	2.0	-1.2258
1	2.0000	2.0	-0.9331
1	2.6000	2.0	-0.7704
1	3.4000	2.0	-0.6929
1	4.4000	2.0	-0.6554
1	5.4000	2.0	-0.6384
1	1.0000	2.5	-5.2346
1	1.2000	2.5	-3.6071
1	1.4000	2.5	-2.6425
1	1.6000	2.5	-2.0560
1	2.0000	2.5	-1.4579
1	2.4000	2.5	-1.1924
1	3.0000	2.5	-1.0100
1	3.8000	2.5	-0.9051
1	4.6000	2.5	-0.8553
1	5.4000	2.5	-0.8276

TABLE XIV

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_0-Direction and Simply Supported and Free on the Two Edges Normal to the y_0-Direction (for K_{y_0} equal to zero only)

m	a_0/b_0	K_{y_0}	K_{x_0}
1	0.6000	0.0	2.7778
1	0.8000	0.0	1.5625
1	1.0000	0.0	1.0000
1	1.2000	0.0	0.6944
1	1.4000	0.0	0.5102
1	1.6000	0.0	0.3906
1	1.8000	0.0	0.3086
1	2.0000	0.0	0.2500
1	2.8000	0.0	0.1276
1	3.6000	0.0	0.0772
1	4.6000	0.0	0.0473
1	5.4000	0.0	0.0343

TABLE XV

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Simply Supported and Free on the Two Edges Normal to the y_o-Direction (for K_{y_o} greater than zero only)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	-5.0	4.0464
1	1.0000	-5.0	2.2947
1	1.4000	-5.0	1.8347
1	1.7225*	-5.0	1.6852
2	2.2000	-5.0	1.1714
2	2.6000	-5.0	0.9422
2	3.2000	-5.0	0.7477
2	3.8000	-5.0	0.6389
2	4.4000	-5.0	0.5721
2	4.9456*	-5.0	0.5315
3	5.2000	-5.0	0.4968
3	5.4000	-5.0	0.4729
1	0.6000	-3.0	3.5453
1	1.0000	-3.0	1.7898
1	1.4000	-3.0	1.3227
1	1.8000	-3.0	1.1408
1	2.1731*	-3.0	1.0588
2	2.6000	-3.0	0.8068
2	3.2000	-3.0	0.6090
2	3.8000	-3.0	0.4977
2	4.6000	-3.0	0.4117
2	5.4000	-3.0	0.3612
1	0.6000	-1.0	3.0416
1	0.8000	-1.0	1.8331
1	1.2000	-1.0	0.9756
1	1.6000	-1.0	0.6788
1	2.2000	-1.0	0.5009
1	3.0000	-1.0	0.4094
1	3.6515*	-1.0	0.3750
2	4.2000	-1.0	0.3020
2	4.8000	-1.0	0.2490
2	5.4000	-1.0	0.2127

TABLE XVI

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported in the x_o-Direction and Simply Supported and Free on the Two Edges Normal to the y_o-Direction

m	a_o/b_o	K_{y_o}	K_{x_o}
1	1.4000	-5.0	1.8347
1	1.4000	-3.0	1.3227
1	1.4000	-1.0	0.7953
1	1.4000	0.0	0.5102
1	1.4000	0.5	0.2926
1	1.4000	1.0	-0.0885
1	1.4000	1.5	-0.6822
1	1.4000	2.0	-1.5272
1	1.4000	2.5	-2.6425
2	3.0000	-5.0	0.7995
2	3.0000	-3.0	0.6619
1	3.0000	-1.0	0.4094
1	3.0000	0.0	0.1111
1	3.0000	0.5	-0.0539
1	3.0000	1.0	-0.2463
1	3.0000	1.5	-0.4683
1	3.0000	2.0	-0.7221
1	3.0000	2.5	-1.0100
3	5.4000	-5.0	0.4729
2	5.4000	-3.0	0.3612
2	5.4000	-1.0	0.2127
1	5.4000	0.0	0.0343
1	5.4000	0.5	-0.1216
1	5.4000	1.0	-0.2856
1	5.4000	1.5	-0.4578
1	5.4000	2.0	-0.6384
1	5.4000	2.5	-0.8276

$$
\begin{aligned}
&[\, Ze^{2cb_o} - Ve^{-2cb_o} \,]\{ (sb_o)^5 + 2(cb_o)^2(sb_o)^3 + (cb_o)^4(sb_o) \\
&\quad - K_{y_o}(\pi b_o/a_o)^2 ((cb_o)^2(sb_o) + (sb_o)^3) \} \\
&\quad + [\, 10(S - R)(cb_o)^2(sb_o)^3 + 5(R - S)(cb_o)^4(sb_o) \\
&\quad + (R - S)(sb_o)^5 + 8(RS - VZ + 1)(cb_o)^3(sb_o)^2 \\
&\quad + 4(VZ - RS - 1)(cb_o)(sb_o)^4 \\
&\quad + K_{y_o}(\pi b_o/a_o)^2 \{ 2(RS - VZ + 1)(cb_o)(sb_o)^2 \\
&\quad + 3(R - S)(cb_o)^2(sb_o) + (S - R)(sb_o)^3 \} \,] \\
&+ 2\sin^2(sb_o)\, [\, (VZ - RS)(cb_o)(sb_o)^4 + (VZ - RS)(cb_o)^5 \\
&\quad + 2(cb_o)^3(sb_o)^2 + K_{y_o}(\pi b_o/a_o)^2 \{ (VZ - RS)(cb_o)^3 \\
&\quad + (cb_o)(sb_o)^2 \} \,] \\
&+ 2\cos^2(sb_o)\, [\, 2(RS - VZ)(cb_o)^3(sb_o)^2 - (cb_o)(sb_o)^4 \\
&\quad - (cb_o)^5 + K_{y_o}(\pi b_o/a_o)^2 \{ (RS - VZ)(cb_o)(sb_o)^2 \\
&\quad - (cb_o)^3 \} \,] \\
&- 2(R + S)\sin(sb_o)\cos(sb_o)\, [\, (cb_o)(sb_o)^4 \\
&\quad + 2(cb_o)^3(sb_o)^2 + (cb_o)^5 \\
&\quad + K_{y_o}(\pi b_o/a_o)^2 \{ (cb_o)(sb_o)^2 + (cb_o)^3 \} \,] = 0 \qquad (350)
\end{aligned}
$$

Equation (350) constitutes the governing equation for any plate aspect ratio and K_{y_o} greater than or equal to zero. In other words, the roots of equation (350) yield the smallest values of K_{x_o} for any a_o/b_o and compressive (or zero) K_{y_o}. Consideration of results generated by equation (350) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

$\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} = 0$

For the quantity $\{ (K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1 \} = 0$

$$
\begin{aligned}
&\sin^2(sb_o)\ [e^{2cb_o}\ \{2Z(cb_o)^2 (sb_o)^3 + Z(cb_o)^4 (sb_o) \\
&\quad + Z(sb_o)^5 - K_{y_o} (\Pi b_o/a_o)^2 (Z(cb_o)^2 (sb_o) + Z(sb_o)^3) \} \\
&\quad - e^{-2cb_o}\ \{V(sb_o)^5 + 2V(cb_o)^2 (sb_o)^3 + V(cb_o)^4 (sb_o) \\
&\quad - K_{y_o} (\Pi b_o/a_o)^2 (V(cb_o)^2 (sb_o) + V(sb_o)^3) \} \\
&\quad + \{8(RS - VZ)(cb_o)^3 (sb_o)^2 + 10(S - R)(cb_o)^2 (sb_o)^3 \\
&\quad + (R - S)(sb_o)^5 + 6(VZ - RS)(cb_o)(sb_o)^4 \\
&\quad + 2(VZ - RS)(cb_o)^5 + 5(R - S)(cb_o)^4 (sb_o) \\
&\quad - 4(cb_o)(sb_o)^4 + 12(cb_o)^3 (sb_o)^2 \\
&\quad + K_{y_o} (\Pi b_o/a_o)^2 (2(RS - VZ)(cb_o)(sb_o)^2 + (S - R)(sb_o)^3 \\
&\quad + 2(VZ - RS)(cb_o)^3 + 3(R - S)(cb_o)^2 (sb_o) \\
&\quad + 4(cb_o)(sb_o)^2) \}] \\
&+ \cos^2(sb_o)\ [e^{2cb_o}\ \{Z(sb_o)^5 + 2Z(cb_o)^2 (sb_o)^3 \\
&\quad + Z(cb_o)^4 (sb_o) - K_{y_o} (\Pi b_o/a_o)^2 (Z(sb_o)^3 \\
&\quad + Z(cb_o)^2 (sb_o)) \} \\
&\quad - e^{-2cb_o}\ \{V(sb_o)^5 + 2V(cb_o)^2 (sb_o)^3 + V(cb_o)^4 (sb_o) \\
&\quad - K_{y_o} (\Pi b_o/a_o)^2 (V(cb_o)^2 (sb_o) + V(sb_o)^3) \} \\
&\quad + \{12(RS - VZ)(cb_o)^3 (sb_o)^2 + 10(S - R)(cb_o)^2 (sb_o)^3 \\
&\quad + (R - S)(sb_o)^5 + 4(VZ - RS)(cb_o)(sb_o)^4 \\
&\quad + 5(R - S)(cb_o)^4 (sb_o) - 6(cb_o)(sb_o)^4 + 8(cb_o)^3 (sb_o)^2 \\
&\quad - 2(cb_o)^5 + K_{y_o} (\Pi b_o/a_o)^2 (4(RS - VZ)(cb_o)(sb_o)^2 \\
&\quad + (S - R)(sb_o)^3 + 3(R - S)(cb_o)^2 (sb_o) \\
&\quad + 2(cb_o)(sb_o)^2 - 2(cb_o)^3) \}] \\
&- 2\sin(sb_o)\cos(sb_o)\ [(R + S)(cb_o)(sb_o)^4 \\
&\quad + 2(R + S)(cb_o)^3 (sb_o)^2 + (R + S)(cb_o)^5 \\
&\quad + K_{y_o} (\Pi b_o/a_o)^2 \{(R + S)(cb_o)(sb_o)^2 \\
&\quad + (R + S)(cb_o)^3 \}] = 0 \qquad (349)
\end{aligned}
$$

$$\begin{vmatrix} \begin{array}{l} \sin(sb_o)\ \{(Sc^2 - Ss^2 \\ \quad - 2cs)e^{cb_o} + (Vc^2 \\ \quad - Vs^2)e^{-cb_o}\ \} \\ + \cos(sb_o)\ \{(2Scs + c^2 \\ \quad - s^2)e^{cb_o} \\ \quad - 2Vcse^{-cb_o}\ \} \end{array} & \begin{array}{l} \sin(sb_o)\ \{(Rc^2 - Rs^2 \\ \quad + 2cs)e^{-cb_o} + (Zc^2 \\ \quad - Zs^2)e^{cb_o}\ \} \\ + \cos(sb_o)\ \{2Zcse^{cb_o} \\ \quad + (c^2 - s^2 \\ \quad - 2Rcs)e^{-cb_o}\ \} \end{array} \\ \hline \begin{array}{l} \sin(sb_o)\ \{(Sc^3 - 3Scs^2 \\ \quad + s^3 - 3c^2 s \\ \quad + K_{y_o}(\pi/a_o)^2\,[Sc \\ \quad - s]\ )e^{cb_o} \\ \quad + (3Vcs^2 - Vc^3 \\ \quad - K_{y_o}(\pi/a_o)^2\,Vc)e^{-cb_o}\ \} \\ + \cos(sb_o)\ \{(3Sc^2 s - Ss^3 \\ \quad + c^3 - 3cs^2 \\ \quad + K_{y_o}(\pi/a_o)^2\,[Ss \\ \quad + c]\ )e^{cb_o} \\ \quad + (3Vc^2 s - Vs^3 \\ \quad + K_{y_o}(\pi/a_o)^2\,Vs)e^{-cb_o}\ \} \end{array} & \begin{array}{l} \sin(sb_o)\ \{(Zc^3 - 3Zcs^2 \\ \quad + K_{y_o}(\pi/a_o)^2\,Zc)e^{cb_o} \\ \quad + (3Rcs^2 - Rc^3 + s^3 \\ \quad - 3c^2 s \\ \quad - K_{y_o}(\pi/a_o)^2\,[Rc \\ \quad + s]\ )e^{-cb_o}\ \} \\ + \cos(sb_o)\ \{(3Zc^2 s - Zs^3 \\ \quad + K_{y_o}(\pi/a_o)^2\,Zs)e^{cb_o} \\ \quad + (3Rc^2 s - Rs^3 + 3cs^2 \\ \quad - c^3 \\ \quad + K_{y_o}(\pi/a_o)^2\,[Rs \\ \quad - c]\ )e^{-cb_o}\ \} \end{array} \end{vmatrix} = 0 \qquad (348)$$

Expansion of the determinant and multiplication by the quantity b_o^5 results in the following equations (the second equation a simplification of the first):

$$Y'(y_o) = B_m\{ [(Sc - s)e^{cy_o} - Vce^{-cy_o}] \sin(sy_o) + [(Ss + c)e^{cy_o} + Vse^{-cy_o}] \cos(sy_o) \} + D_m\{ [Zce^{cy_o} - (Rc + s)e^{-cy_o}] \sin(sy_o) + [Zse^{cy_o} + (Rs - c)e^{-cy_o}] \cos(sy_o) \} \quad (345)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = B_m\{ [(Sc^2 - Ss^2 - 2cs)e^{cy_o} + (Vc^2 - Vs^2)e^{-cy_o}] \sin(sy_o) + [(2Scs + c^2 - s^2)e^{cy_o} - 2Vcse^{-cy_o}] \cos(sy_o) \} + D_m\{ [(Zc^2 - Zs^2)e^{cy_o} + (Rc^2 - Rs^2 - 2cs)e^{-cy_o}] \sin(sy_o) + [2Zcse^{-cy_o} + (c^2 - s^2 - 2Rcs)e^{cy_o}] \cos(sy_o) \} \quad (346)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = B_m\{ [(Sc^3 - 3Scs^2 + s^3 - 3c^2s)e^{cy_o} + (3Vcs^2 - Vc^3)e^{-cy_o}] \sin(sy_o) + [(3Sc^2s - Ss^3 + c^3 - 3cs^2)e^{cy_o} + (3Vc^2s - Vs^3)e^{-cy_o}] \cos(sy_o) \} + D_m\{ [(Zc^3 - 3Zcs^2)e^{cy_o} + (3Rcs^2 - Rc^3 + s^3 - 3c^2s)e^{-cy_o}] \sin(sy_o) + [(3Zc^2s - Zs^3)e^{cy_o} + (3Rc^2s - Rs^3 + 3cs^2 - c^3)e^{-cy_o}] \cos(sy_o) \} \quad (347)$$

Substitution of equations (345), (346), and (347) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients B_m and D_m. For non-trivial B_m and D_m, the following determinental equation must hold:

$$V = \{ -0.5\ s^4/c - cs^2 - 0.5\ c^3 + K_{y_o} (\pi/a_o)^2 [0.5\ c + 0.5\ s^2/c] \} / \{ 2s^3 - 6c^2 s - 2\ K_{y_o} (\pi/a_o)^2 s \} \quad (338)$$

$$R = \{ -0.5\ s^4/c + 5cs^2 - 2.5\ c^3 + K_{y_o} (\pi/a_o)^2 \cancel{[-1.5}\ c + 0.5\ s^2/c] \} / \{ 2s^3 - 6\ c^2 s - 2\ K_{y_o} (\pi/a_o)^2 s \} \quad (339)$$

Therefore, equation (337) can be written more efficiently in terms of the variables V and R.

$$C_m = V\ B_m + R\ D_m \quad (340)$$

In a similar fashion, define two additional variables of convenience:

$$S = V - (c^2 - s^2)/(2cs) \quad (341)$$

$$Z = R - (c^2 - s^2)/(2cs) \quad (342)$$

With knowledge of the variables V and R, S and Z allow equation (335) to be rewritten in a concise form.

$$A_m = S\ B_m + Z\ D_m \quad (343)$$

Finally, the $Y(y_o)$ function, equation (106), can be written in terms of the constants B_m and D_m only when equations (340) and (343) are substituted into equation (106).

$$Y(y_o) = B_m\{ (Se^{cy_o} + Ve^{-cy_o}) \sin(sy_o) + e^{cy_o} \cos(sy_o) \} + D_m\{ (Ze^{cy_o} + Re^{-cy_o}) \sin(sy_o) + e^{-cy_o} \cos(sy_o) \} \quad (344)$$

It is necessary to differentiate this $Y(y_o)$ function again. Only two constants, as opposed to four, are now present in this function. The first derivative of $Y(y_o)$ with respect to y_o is:

$$
\begin{aligned}
Y'''(y_o) = {} & A_m e^{cy_o} \{ (c^3 - 3cs^2) \sin(sy_o) \\
& \quad + (3c^2 s - s^3) \cos(sy_o) \} \\
& + B_m e^{cy_o} \{ (s^3 - 3c^2 s) \sin(sy_o) \\
& \quad + (c^3 - 3cs^2) \cos(sy_o) \} \\
& + C_m e^{-cy_o} \{ (3cs^2 - c^3) \sin(sy_o) \\
& \quad + (3c^2 s - s^3) \cos(sy_o) \} \\
& + D_m e^{-cy_o} \{ (s^3 - 3c^2 s) \sin(sy_o) \\
& \quad + (3cs^2 - c^3) \cos(sy_o) \}
\end{aligned} \tag{333}
$$

Apply equation (327) to equation (332). This stipulation fixes A_m in terms of B_m, C_m, and D_m.

$$A_m\{2cs\} + B_m\{ c^2 - s^2 \} - C_m\{2cs\} + D_m\{ c^2 - s^2 \} = 0 \tag{334}$$

$$A_m = C_m - [(c^2 - s^2)/(2cs)] \{ B_m + D_m \} \tag{335}$$

Next, apply equation (328) to the combination of equations (331) and (333). This condition will fix C_m (and hence A_m by equation (335)) in terms of B_m and D_m.

$$
\begin{aligned}
& A_m\{3c^2 s - s^3 + K_{y_o} (\Pi/a_o)^2 s \} + B_m\{c^3 - 3cs^2 \\
& + K_{y_o} (\Pi/a_o)^2 c\} + C_m\{3c^2 s - s^3 + K_{y_o} (\Pi/a_o)^2 s \} \\
& \quad + D_m\{3cs^2 - c^3 - K_{y_o} (\Pi/a_o)^2 c \} = 0
\end{aligned} \tag{336}
$$

When the relation for A_m expressed in equation (335) is inserted into equation (336), the following expression is returned:

$$
\begin{aligned}
C_m = {} & [B_m\{ -0.5\, s^4/c - cs^2 - 0.5\, c^3 + K_{y_o} (\Pi/a_o)^2 (0.5\, c \\
& + 0.5\, s^2/c) \} + D_m\{ -0.5\, s^4/c + 5cs^2 - 2.5\, c^3 \\
& + K_{y_o} (\Pi/a_o)^2 (-1.5\, c + 0.5\, s^2/c) \}]/[2s^3 \\
& \quad - 6c^2 s - 2\, K_{y_o} (\Pi/a_o)^2 s]
\end{aligned} \tag{337}
$$

Define the following variables:

$Y''(b_o)\ \sin(m\pi x_o/a_o) = 0\ ;$

$$[Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o)]\ \sin(m\pi x_o/a_o) = 0 \tag{326}$$

For equations (325) and (326) to have meaning in the general case, the following conditions must hold:

$$Y''(0) = 0 \tag{327}$$

$$Y'''(0) + K_{y_o}(\pi/a_o)^2 Y'(0) = 0 \tag{328}$$

$$Y''(b_o) = 0 \tag{329}$$

$$Y'''(b_o) + K_{y_o}(\pi/a_o)^2 Y'(b_o) = 0 \tag{330}$$

Again, the function $Y(y_o)$ is expressed in equation (106). The first derivative of this function with respect to y_o is:

$$\begin{aligned} Y'(y_o) = {} & A_m e^{cy_o}\{\ s\cos(sy_o) + c\sin(sy_o)\ \} \\ & - B_m e^{cy_o}\{\ s\sin(sy_o) - c\cos(sy_o)\ \} \\ & + C_m e^{-cy_o}\{\ s\cos(sy_o) - c\sin(sy_o)\ \} \\ & - D_m e^{-cy_o}\{\ s\sin(sy_o) + c\cos(sy_o)\ \} \end{aligned} \tag{331}$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y''(y_o) = {} & A_m e^{cy_o}\{(c^2 - s^2)\sin(sy_o) + 2cs\cos(sy_o)\ \} \\ & + B_m e^{cy_o}\{(c^2 - s^2)\cos(sy_o) - 2cs\sin(sy_o)\ \} \\ & + C_m e^{-cy_o}\{(c^2 - s^2)\sin(sy_o) - 2cs\cos(sy_o)\ \} \\ & + D_m e^{-cy_o}\{(c^2 - s^2)\cos(sy_o) + 2cs\sin(sy_o)\ \} \end{aligned} \tag{332}$$

The third derivative of $Y(y_o)$ with respect to y_o is:

In the same pattern as in prior sections, a displacement function w which satisfies the first two stipulations of equation (324) is given by equation (77). Furthermore, substitution of this relation for w into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable r is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states--negative, zero, and positive--of the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0$ equations (82) through (105) illustrate that the unknown function $Y(y_o)$ must take the form shown in equation (106).

Consider now the boundary conditions, equations (324), for this case of a laminate simply supported on one set of opposite sides and free on the other. When the chosen form of w, equation (77), is substituted into the final four lines of equations (324), the following must hold:

$$Y''(0)\sin(m\pi x_o/a_o) = 0\ ;$$

$$[Y'''(0) + K_{y_o}(\pi/a_o)^2 Y'(0)]\sin(m\pi x_o/a_o) = 0 \tag{325}$$

VIII. Flat Rectangular Composite Laminate Simply Supported in the x_o-Direction and Free in the y_o-Direction

In contrast to the work presented in the last three sections, the boundary conditions for a laminate simply supported in the x_o-direction and free in the y_o-direction are symmetric with respect to each planar axis. For the edges which have normals parallel to the x_o-axis, the vertical displacement along each edge and the normal component of the moment to each edge must be zero in the affine space. On the other hand, for the two edges which have normals parallel to the y_o-axis, the vanishing of the normal component of the moment to each edge and the satisfaction of equation (224) constitute the two requisites. In equation form, the following must hold:

$$
\begin{aligned}
&\text{on edge } x_o=-a_o/2\ , \quad w = 0 \ ; \ w_{,x_o x_o} = 0 \\
&\text{on edge } x_o= a_o/2\ , \quad w = 0 \ ; \ w_{,x_o x_o} = 0 \\
&\text{on edge } y_o= 0\ , \quad w_{,y_o y_o} = 0 \ ; \\
&\qquad w_{,y_o y_o y_o} + K_{y_o}(\pi/a_o)^2 w_{,y_o} = 0 \\
&\text{on edge } y_o= b_o\ , \quad w_{,y_o y_o} = 0 \ ; \\
&\qquad w_{,y_o y_o y_o} + K_{y_o}(\pi/a_o)^2 w_{,y_o} = 0
\end{aligned}
\qquad (324)
$$

Note that the origin of coordinates in the affine space is taken to be at the center of one of the free edges normal to the y_o-direction. This choice of origin location, despite the symmetric boundary conditions, affords one maximum simplicity in numerical manipulations.

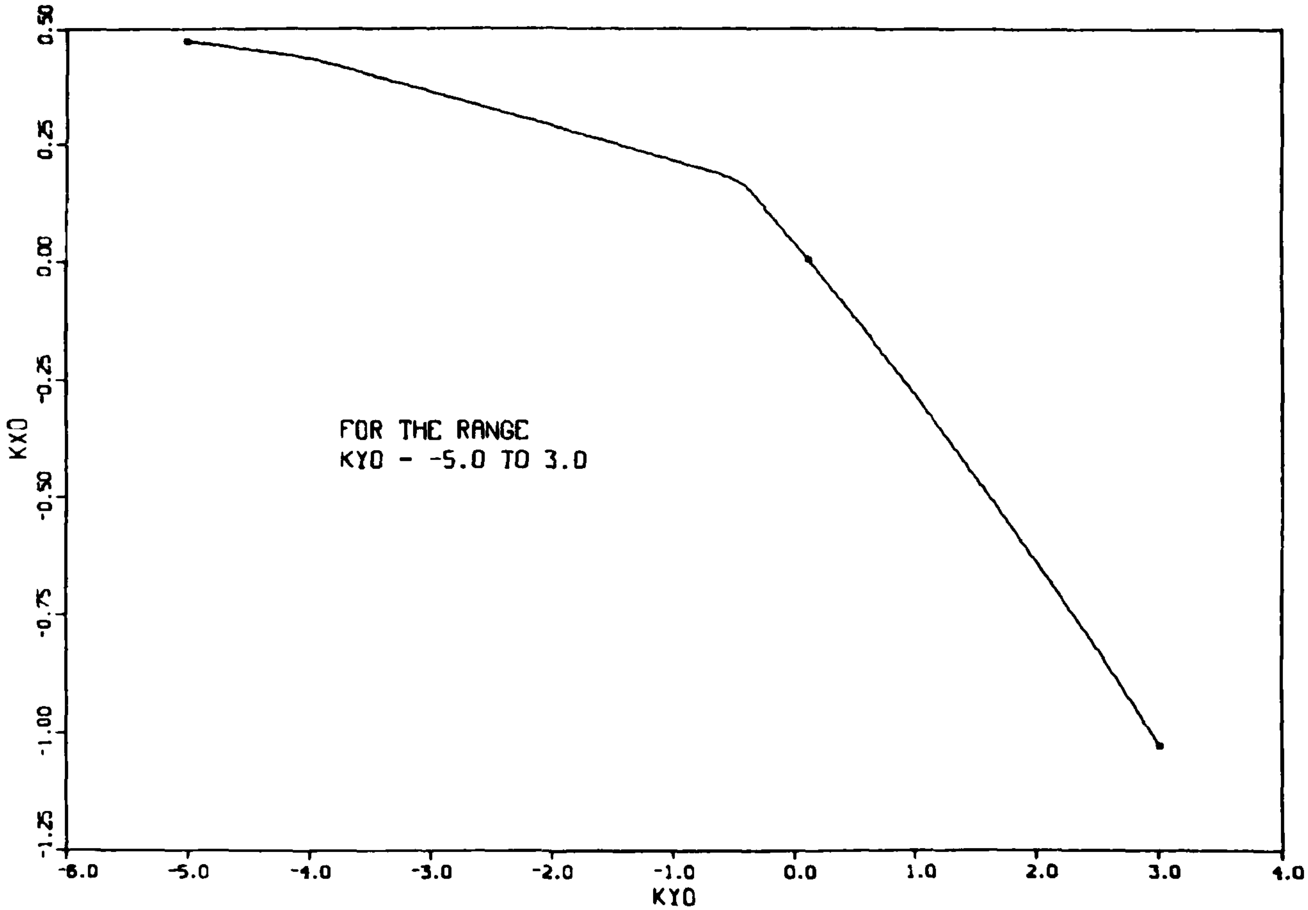


FIGURE 31

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 5.4 FOR AN S-S-S-F LAMINATE

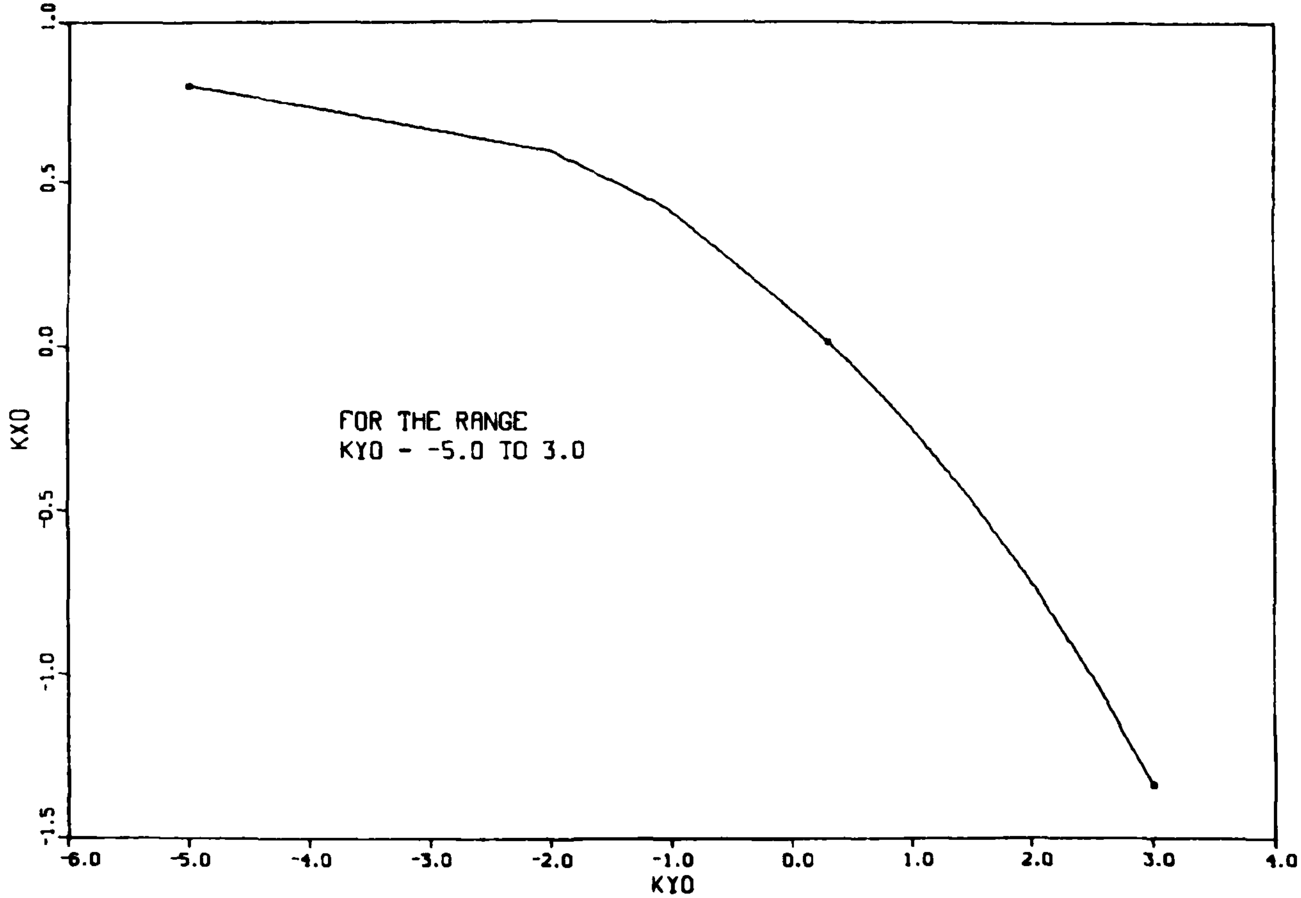


FIGURE 30

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 3.0 FOR AN S-S-S-F LAMINATE

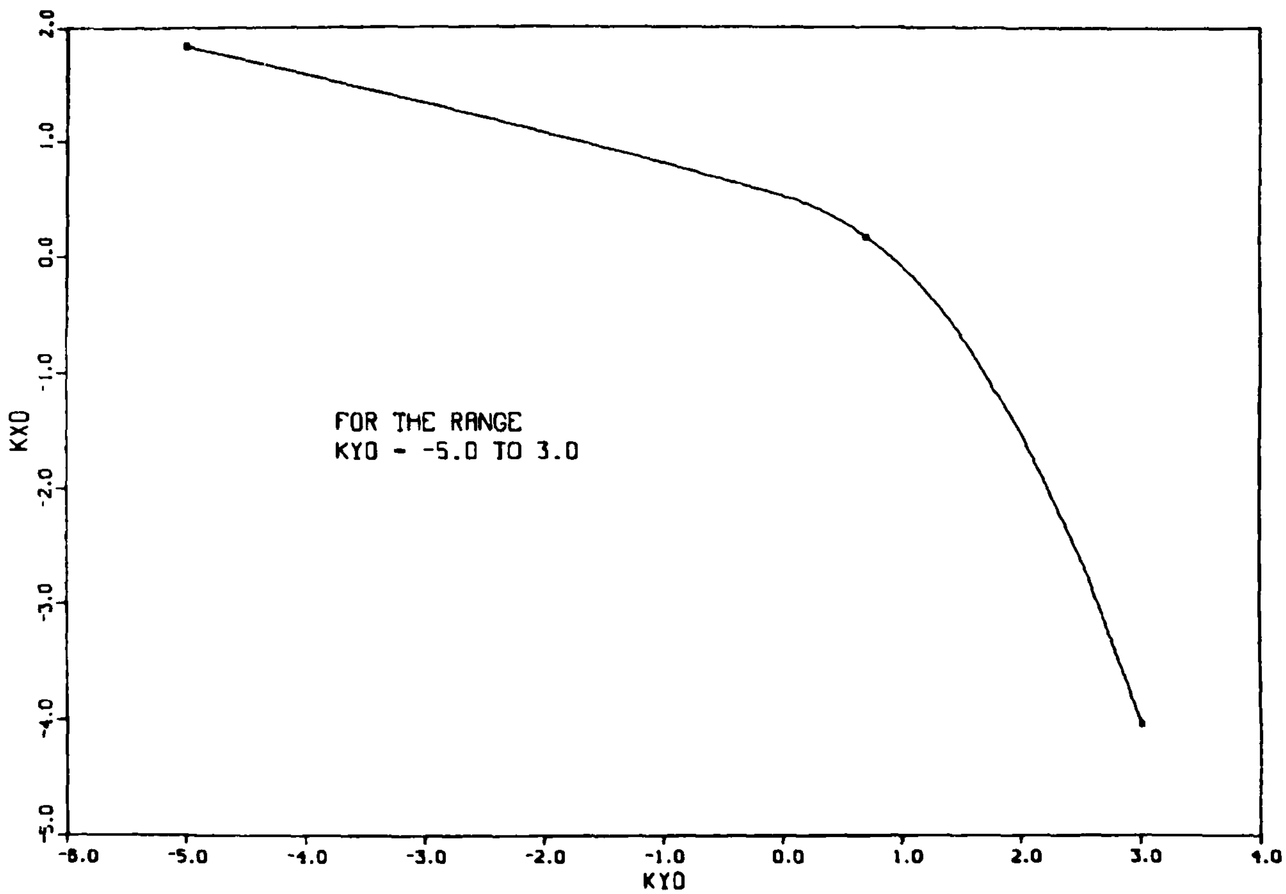
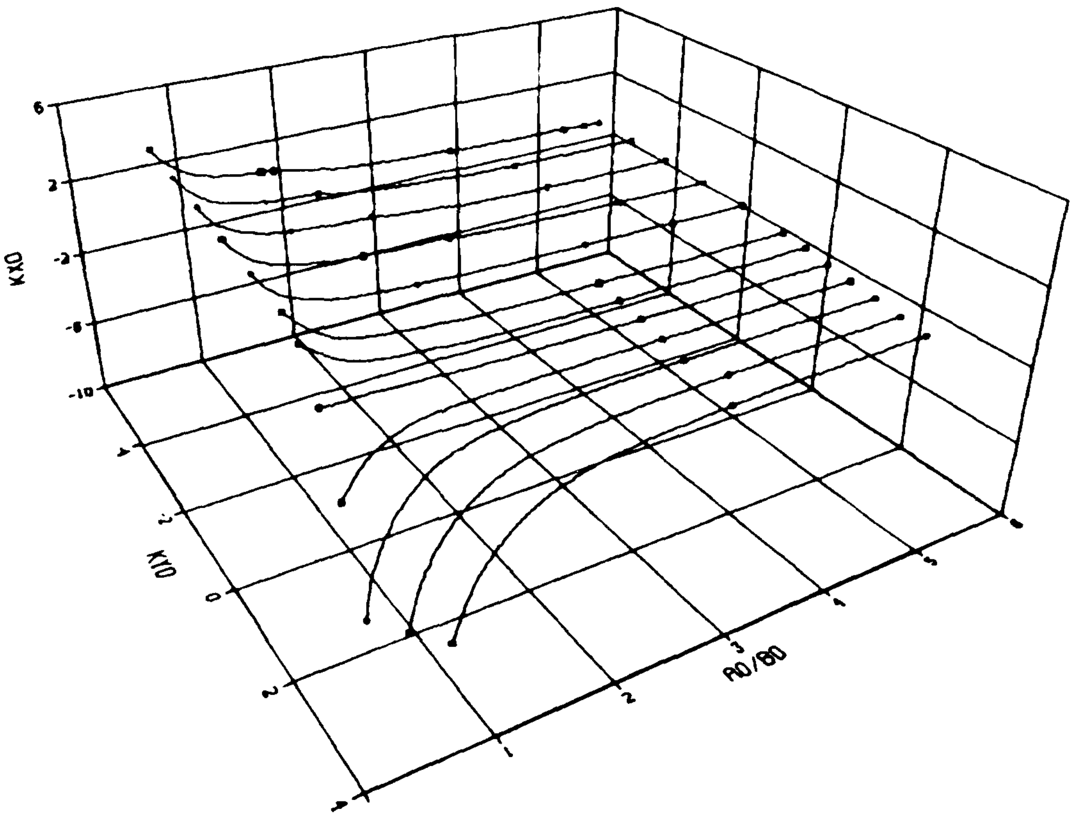


FIGURE 29

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.4 FOR AN S-S-S-F LAMINATE



FIGURF 2A

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-S-S-F LAMINATE

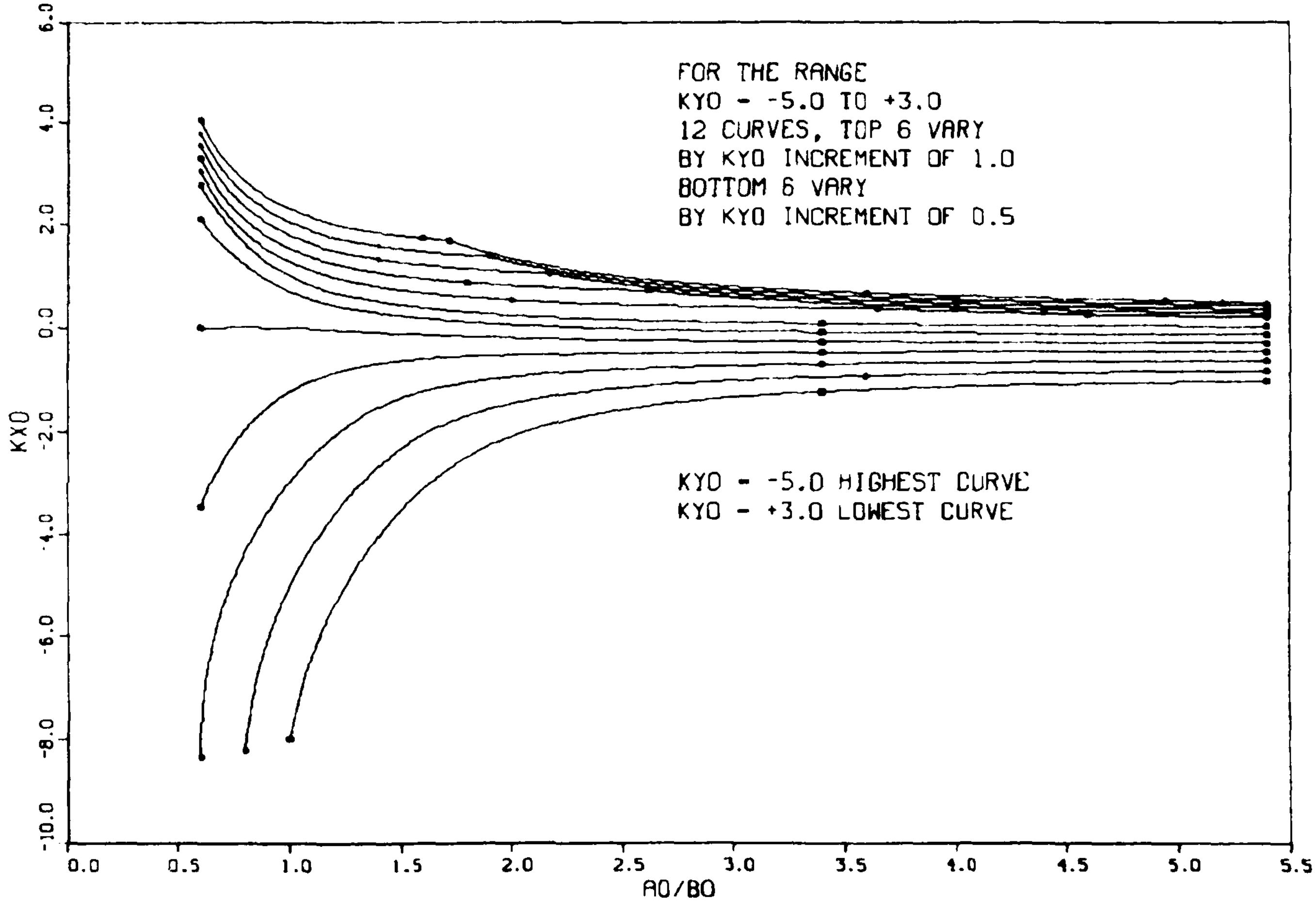


FIGURE 27

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-S-S-F LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of K_{y_o}, each possible range of K_{y_o} --negative, zero, and positive--will be analyzed as different subcases.

$K_{y_o} < 0$.

For $K_{y_o} < 0$ equations (121) and (122) lead to the conclusion that the unknown $Y(y_o)$ function must take the form shown in equation (123). The first derivative of this function with respect to y_o is:

$$\begin{aligned} Y'(y_o) = {} & A_m T \cosh(Ty_o) + B_m T \sinh(Ty_o) \\ & + C_m\{ \sinh(Ty_o) + Ty_o \cosh(Ty_o) \} \\ & + D_m\{ \cosh(Ty_o) + Ty_o \sinh(Ty_o) \} \end{aligned} \tag{351}$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y''(y_o) = {} & A_m T^2 \sinh(Ty_o) + B_m T^2 \cosh(Ty_o) \\ & + C_m\{ 2T \cosh(Ty_o) + T^2 y_o \sinh(Ty_o) \} \\ & + D_m\{ 2T \sinh(Ty_o) + T^2 y_o \cosh(Ty_o) \} \end{aligned} \tag{352}$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y'''(y_o) = {} & A_m T^3 \cosh(Ty_o) + B_m T^3 \sinh(Ty_o) \\ & + C_m\{ 3T^2 \sinh(Ty_o) + T^3 y_o \cosh(Ty_o) \} \\ & + D_m\{ 3T^2 \cosh(Ty_o) + T^3 y_o \sinh(Ty_o) \} \end{aligned} \tag{353}$$

Apply equation (327) to equation (352). This combination fixes C_m in terms of B_m such that $C_m = -(T/2)B_m$ Next, couple equations (351) and (353) so that they adhere to the boundary condition expressed in equation (328). D_m is therefore related to A_m in the following manner:

$$D_m = -A_m \{ T^3 + K_{y_o} (\pi/a_o)^2 T \} / \{ 3T^2 + K_{y_o} (\pi/a_o)^2 \} \tag{354}$$

Define the following variable:

$$H = - \{ T^3 + K_{y_o} (\pi/a_o)^2 T \} / \{ 3T^2 + K_{y_o} (\pi/a_o)^2 \} \qquad (355)$$

With the variable H so defined, the $Y(y_o)$ function, equation (123), can be written in terms of the constants A_m and B_m only.

$$Y(y_o) = A_m\{ \sinh(Ty_o) + Hy_o \cosh(Ty_o) \}$$
$$+ B_m\{ \cosh(Ty_o) - (T/2)\, y_o \sinh(Ty_o) \} \qquad (356)$$

As explained in the previous paragraphs, it is now necessary to differentiate this slimmed down expression three times. The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{(T + H) \cosh(Ty_o) + HT\, y_o \sinh(Ty_o)\}$$
$$+ B_m\{ (T/2) \sinh(Ty_o) - (T^2/2)\, y_o \cosh(Ty_o)\} \qquad (357)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m\{(T^2 + 2HT) \sinh(Ty_o) + T^2H\, y_o \cosh(Ty_o)\}$$
$$- B_m\{(T^3/2)\, y_o \sinh(Ty_o) \} \qquad (358)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{(T^3 + 3T^2H) \cosh(Ty_o) + T^3H\, y_o \sinh(Ty_o)\}$$
$$- B_m\{(T^3/2) \sinh(Ty_o) + (T^4/2) y_o \cosh(Ty_o)\} \qquad (359)$$

Substitution of equations (357), (358), and (359) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sinh(Tb_o)\ \{T^2 + 2TH\} \\ + \cosh(Tb_o)\ \{T^2 Hb_o\} \end{array} & \begin{array}{l} \sinh(Tb_o)\ \{-T^3 b_o/2\} \end{array} \\ \begin{array}{l} \sinh(Tb_o)\ \{T^3 Hb_o \\ \quad + K_{y_o} (\pi/a_o)^2\, HTb_o\} \\ + \cosh(Tb_o)\ \{T^3 + 3T^2 H \\ \quad + K_{y_o} (\pi/a_o)^2\, (T + H)\} \end{array} & \begin{array}{l} \sinh(Tb_o)\ \{-T^3/2 \\ \quad + K_{y_o} (\pi/a_o)^2\, (T/2)\ \} \\ + \cosh(Tb_o)\ \{-T^4 b_o/2 \\ \quad - K_{y_o} (\pi/a_o)^2\, (T^2 b_o/2)\} \end{array} \end{vmatrix} = 0 \qquad (360)$$

Expansion of the determinant and multiplication by the quantity b_o^5 results in the following equations (the second a simplification of the first):

$$\begin{aligned} &\sinh^2(Tb_o)\ [\ 0.5\ (Hb_o)(Tb_o)^6 - 0.5\ (Tb_o)^5 - (Hb_o)(Tb_o)^4 \\ &\quad + K_{y_o} (\pi b_o/a_o)^2 \{\ 0.5\ (Tb_o)^3 + (Hb_o)(Tb_o)^2 \\ &\quad + 0.5\ (Hb_o)(Tb_o)^4\ \}\ ] \\ &- \cosh^2(Tb_o)\ [\ 0.5\ (Hb_o)(Tb_o)^6 \\ &\quad + K_{y_o} (\pi b_o/a_o)^2 \{\ 0.5\ (Hb_o)(Tb_o)^4\ \}\ ] = 0 \end{aligned} \qquad (361)$$

$$\begin{aligned} &\sinh^2(Tb_o)\ [\ - (Tb_o)^5 - 2(Hb_o)(Tb_o)^4 \\ &\quad + K_{y_o} (\pi b_o/a_o)^2 \{\ (Tb_o)^3 + 2(Hb_o)(Tb_o)^2\ \}\ ] \\ &\quad - (Hb_o)(Tb_o)^6 - K_{y_o} (\pi b_o/a_o)^2 \{\ (Hb_o)(Tb_o)^4\ \} = 0 \end{aligned} \qquad (362)$$

No value of (Tb_o) greater than zero can satisfy equation (362). *Therefore, no possible solutions exist for the* present boundary conditions for

$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ and $K_{y_o} < 0$

$K_{y_o} = 0$.

For $K_{y_o} = 0$ equation (128) constitutes the required shape of the unknown $Y(y_o)$ function. In addition, equations (327) through (330) are the sets of constraints for this

$Y(y_o)$ function. Moreover, notice that equations (328) and (330) reduce to $Y'''(0) = 0$ and $Y'''(b_o) = 0$ respectively because $K_{y_o} = 0$ Equations (327) and (328) imply that $C_m = D_m = 0$ After the imposition of these two conditions, equation (128) reduces to:

$$Y(y_o) = A_m + B_m y_o \qquad (363)$$

Equations (329) and (330) are automatically satisfied for this $Y(y_o)$ given by equation (363). All boundary conditions, equations (327) through (330), are therefore upheld by this $Y(y_o)$. As a result, if $K_{y_o} = 0$ equation (298) is a valid solution for K_{x_o}. However, equation (298) does not yield the minimum K_{x_o} for any combination of plate aspect ratio and $K_{y_o} = 0$ Consequently, further consideration of this subcase is dropped.

$K_{y_o} > 0$.

For $K_{y_o} > 0$ equations (130) and (131) show that the unknown function $Y(y_o)$ must fit the relation given by equation (132). The first derivative of this function with respect to y_o is:

$$\begin{aligned} Y'(y_o) = {} & A_m U \cos(Uy_o) - B_m U \sin(Uy_o) \\ & + C_m\{ \sin(Uy_o) + Uy_o \cos(Uy_o) \} \\ & + D_m\{ \cos(Uy_o) - Uy_o \sin(Uy_o) \} \qquad (364) \end{aligned}$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$\begin{aligned} Y''(y_o) = {} & - A_m U^2 \sin(Uy_o) - B_m U^2 \cos(Uy_o) \\ & + C_m\{ 2U \cos(Uy_o) - U^2 y_o \sin(Uy_o) \} \\ & - D_m\{ 2U \sin(Uy_o) + U^2 y_o \cos(Uy_o) \} \qquad (365) \end{aligned}$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = -A_m U^3 \cos(Uy_o) + B_m U^3 \sin(Uy_o) - C_m\{ 3U^2 \sin(Uy_o) + U^3 y_o \cos(Uy_o) \} - D_m\{ 3U^2 \cos(Uy_o) - U^3 y_o \sin(Uy_o) \} \quad (366)$$

Apply equation (327) to equation (365). This combination fixes C_m in terms of B_m such that $C_m = (U/2) B_m$ Next, couple equations (364) and (366) so that they adhere to the boundary condition expressed in equation (328). D_m is thus related to A_m in the following manner:

$$D_m = A_m\{ U^3 - K_{y_o} (\Pi/a_o)^2 U \} / \{-3U^2 + K_{y_o} (\Pi/a_o)^2 \} \quad (367)$$

Define the following variable:

$$G = \{ U^3 - K_{y_o} (\Pi/a_o)^2 U \} / \{-3U^2 + K_{y_o} (\Pi/a_o)^2 \} \quad (368)$$

With the variable G so defined, the $Y(y_o)$ function, equation (132), can be written in terms of the constants A_m and B_m only.

$$Y(y_o) = A_m\{ \sin(Uy_o) + G\, y_o \cos(Uy_o) \} + B_m\{ \cos(Uy_o) + (U/2)\, y_o \sin(Uy_o) \} \quad (369)$$

Again, this function must be differentiated three times. The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{(U + G) \cos(Uy_o) - UG\, y_o \sin(Uy_o) \} - B_m\{(U/2) \sin(Uy_o) - (U^2 y_o/2) \cos(Uy_o) \} \quad (370)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m\{(U^2 + 2UG) \sin(Uy_o) + U^2 G\, y_o \cos(Uy_o) \} - B_m\{ (U^3 y_o/2) \sin(Uy_o) \} \quad (371)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{-(U^3 + 3GU^2)\cos(Uy_o) + U^3G\, y_o \sin(Uy_o)\}$$
$$- B_m\{(U^3/2)\sin(Uy_o) + (U^4/2)\, y_o \cos(Uy_o)\} \quad (372)$$

Substitution of equations (370), (371), and (372) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sin(Ub_o)\ \{-U^2 - 2UG\} \\ +\cos(Ub_o)\ \{-U^2Gb_o\} \end{array} & \sin(Ub_o)\ \{-U^3b_o/2\} \\ \begin{array}{l} \sin(Ub_o)\ \{U^3Gb_o \\ \quad - K_{y_o}(\pi/a_o)^2\, UGb_o\} \\ +\cos(Ub_o)\ \{-U^3 - 3U^2G \\ \quad + K_{y_o}(\pi/a_o)^2\, (U + G)\} \end{array} & \begin{array}{l} \sin(Ub_o)\ \{-U^3/2 \\ \quad - K_{y_o}(\pi/a_o)^2\ U/2\} \\ +\cos(Ub_o)\ \{-U^4b_o/2 \\ \quad + K_{y_o}(\pi/a_o)^2\, \{U^2b_o/2\} \end{array} \end{vmatrix} = 0 \quad (373)$$

Expansion of the determinant and multiplication by the quantity b_o^5 results in the following system of equations (the second equation a simplification of the first):

$$\begin{aligned} &\sin^2(Ub_o)\ \{\ 0.5\ (Gb_o)(Ub_o)^6 + 0.5\ (Ub_o)^5 + (Gb_o)(Ub_o)^4 \\ &\quad + K_{y_o}(\pi b_o/a_o)^2\,[\ 0.5\ (Ub_o)^3 + (Gb_o)(Ub_o)^2 \\ &\quad - 0.5\ (Gb_o)(Ub_o)^4\ ]\ \} \\ &+ \cos^2(Ub_o)\ \{\ 0.5\ (Gb_o)(Ub_o)^6 \\ &\quad - K_{y_o}(\pi b_o/a_o)^2\,[\ 0.5\ (Gb_o)(Ub_o)^4\ ]\ \} = 0 \end{aligned} \quad (374)$$

$$\begin{aligned} &\sin^2(Ub_o)\ \{\ (Ub_o)^5 + 2(Gb_o)(Ub_o)^4 \\ &\quad + K_{y_o}(\pi b_o/a_o)^2\,[\ (Ub_o)^3 + 2(Gb_o)(Ub_o)^2\ ]\ \} \\ &+ (Gb_o)(Ub_o)^6 - K_{y_o}(\pi b_o/a_o)^2\,[(Gb_o)(Ub_o)^4\ ] = 0 \end{aligned} \quad (375)$$

No value of (Ub_o) greater than zero can satisfy equation (375). Therefore, no possible solutions exist for the present boundary conditions for

$$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0 \quad \text{and } K_{y_o} > 0$$

$$\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$$

For the quantity $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $Y(y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of K_{y_o} for valid solutions. Three subcases must be considered so that a solution for K_{x_o} may be determined for any range of K_{y_o}.

K_{y_o} <u>Ranges from a Relatively Large Positive Number to Positive Infinity</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, K_{y_o} can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities

expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144).

The first derivative with repect to y_o of this function $Y(y_o)$ of equation (144) is:

$$Y'(y_o) = A_m\alpha_m \cos(\alpha_m y_o) - B_m\alpha_m \sin(\alpha_m y_o) + C_m\gamma_m \cos(\gamma_m y_o) - D_m\gamma_m \sin(\gamma_m y_o) \qquad (376)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = - A_m\alpha_m^2 \sin(\alpha_m y_o) - B_m\alpha_m^2 \cos(\alpha_m y\ ) - C_m\gamma_m^2 \sin(\gamma_m y_o) - D_m\gamma_m^2 \cos(\gamma_m y_o) \qquad (377)$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = - A_m\alpha_m^3 \cos(\alpha_m y_o) + B_m\alpha_m^3 \sin(\alpha_m y_o) - C_m\gamma_m^3 \cos(\gamma_m y_o) + D_m\gamma_m^3 \cos(\gamma_m y_o) \qquad (378)$$

Apply equation (327) to equation (377). This combination fixes D_m in terms of B_m such that $D_m = -(\alpha_m/\gamma_m)^2 B_m$ Next, couple equations (376) and (378) so that they mirror the boundary condition expressed in equation (328). C_m is thus related to A_m in the following manner:

$$C_m = A_m\{ \alpha_m^3 - K_{y_o} (\Pi/a_o)^2 \alpha_m \} / \{-\gamma_m^3 + K_{y_o} (\Pi/a_o)^2 \gamma_m \} \qquad (379)$$

Define the following variable:

$$L = \{ \alpha_m^3 - K_{y_o} (\Pi/a_o)^2 \alpha_m \} / \{-\gamma_m^3 + K_{y_o} (\Pi/a_o)^2 \gamma_m \} \qquad (380)$$

With the variable L so defined, the $Y(y_o)$ function, equation (144), can be written in terms of the constants A_m and B_m only.

$$Y(y_o) = A_m\{ \sin(\alpha_m y_o) + L \sin(\gamma_m y_o) \}$$
$$+ B_m\{ \cos(\alpha_m y_o) - (\alpha_m/\gamma_m)^2 \cos(\gamma_m y_o) \} \qquad (381)$$

For reasons detailed in prior work, equation (381) must be differentiated three times. The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y\ ) = A_m\{\alpha_m \cos(\alpha_m y_o) + L\,\gamma_m \cos(\gamma_m y_o) \}$$
$$- B_m\{\alpha_m \sin(\alpha_m y_o) - (\alpha_m^2/\gamma_m) \sin(\gamma_m y_o) \} \qquad (382)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = - A_m\{\alpha_m^2 \sin(\alpha_m y_o) + L\,\gamma_m^2 \sin(\gamma_m y_o) \}$$
$$- B_m\{\alpha_m^2 \cos(\alpha_m y_o) - \alpha_m^2 \cos(\gamma_m y_o) \} \qquad (383)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = - A_m\{\alpha_m^3 \cos(\alpha_m y_o) + L\,\gamma_m^3 \cos(\gamma_m y_o) \}$$
$$+ B_m\{\alpha_m^3 \sin(\alpha_m y_o) - \alpha_m^2 \gamma_m \sin(\gamma_m y_o) \} \qquad (384)$$

Substitution of equations (382), (383), and (384) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sin(\alpha_m b_o)\ \{-\alpha_m^2\} \\ + \sin(\gamma_m b_o)\ \{-L\gamma_m^2\} \end{array} & \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^2\} \\ + \cos(\gamma_m b_o)\ \{\alpha_m^2\} \end{array} \\ \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^3 \\ \quad + K_{y_o} (\pi/a_o)^2 \alpha_m \} \\ + \cos(\gamma_m b_o)\ \{-L\gamma_m^3 \\ \quad + K_{y_o} (\pi/a_o)^2 L\gamma_m\} \end{array} & \begin{array}{l} \sin(\alpha_m b_o)\ \{\alpha_m^3 \\ \quad - K_{y_o} (\pi/a_o)^2 \alpha_m \} \\ + \sin(\gamma_m b_o)\ \{-\alpha_m^2\gamma_m \\ \quad + K_{y_o} (\pi/a_o)^2 \alpha_m^2/\gamma_m\} \end{array} \end{vmatrix} = 0 \qquad (385)$$

Expansion of the determinant, division by the common

multiple α_m, and multiplication by the quantity $\gamma_m b_o^5$ gives:

$$\sin(\alpha_m b_o)\ \sin(\gamma_m b_o)\ [\ (\alpha_m b_o)^3 (\gamma_m b_o)^2 - L\ (\alpha_m b_o)^2 (\gamma_m b_o)^3$$
$$+ K_{y_o} (\pi b_o/a_o)^2 \{\ L\ (\gamma_m b_o)^3 - (\alpha_m b_o)^3\ \}\ ]$$
$$+ \cos(\alpha_m b_o)\ \cos(\gamma_m b_o)\ [\ (\alpha_m b_o)^4 (\gamma_m b_o) - L\ (\alpha_m b_o)\ (\gamma_m b_o)^4$$
$$+ K_{y_o} (\pi b_o/a_o)^2 \{\ L\ (\alpha_m b_o)(\gamma_m b_o)^2 - (\alpha_m b_o)^2 (\gamma_m b_o)\ \}\ ]$$
$$- (\alpha_m b_o)^4 (\gamma_m b_o) + L\ (\alpha_m b_o)(\gamma_m b_o)^4$$
$$+ K_{y_o} (\pi b_o/a_o)^2 \{\ (\alpha_m b_o)^2 (\gamma_m b_o) - L\ (\alpha_m b_o)(\gamma_m b_o)^2\ \} = 0 \qquad (386)$$

Attempts at solution of equation (386) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase are dropped.

K_{y_o} <u>Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number</u>.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, K_{y_o} can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, K_{y_o} can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that K_{y_o} range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that $Y(y_o)$ must take the form shown in equation (162).

The first derivative with respect to y_o of this function

$Y(y_o)$ in equation (162) is:

$$Y'(y_o) = A_m\alpha_m \cos(\alpha_m y_o) - B_m\alpha_m \sin(\alpha_m y_o) + C_m\beta_m \cosh(\beta_m y_o) + D_m\beta_m \sinh(\beta_m y_o) \quad (387)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = - A_m\alpha_m^2 \sin(\alpha_m y_o) - B_m\alpha_m^2 \cos(\alpha_m y_o) + C_m\beta_m^2 \sinh(\beta_m y_o) + D_m\beta_m^2 \cosh(\beta_m y_o) \quad (388)$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = - A_m\alpha_m^3 \cos(\alpha_m y_o) + B_m\alpha_m^3 \sin(\alpha_m y_o) + C_m\beta_m^3 \cosh(\beta_m y_o) + D_m\beta_m^3 \sinh(\beta_m y_o) \quad (389)$$

Apply equation (327) to equation (388). This combination fixes D_m in terms of B_m such that $D_m = (\alpha_m/\beta_m)^2 B_m$ Next, couple equations (387) and (389) so that they conform to the boundary condition expressed in equation (328). C_m is thus related to A_m in the following manner:

$$C_m = A_m\{ \alpha_m^3 - K_{y_o} (\pi/a_o)^2 \alpha_m \} / \{ \beta_m^3 + K_{y_o} (\pi/a_o)^2 \beta_m \} \quad (390)$$

Define the following variable:

$$J = \{ \alpha_m^3 - K_{y_o} (\pi/a_o)^2 \alpha_m \} / \{ \beta_m^3 + K_{y_o} (\pi/a_o)^2 \beta_m \} \quad (391)$$

With the variable J so defined, the $Y(y_o)$ function, equation (162), can be written in terms of the constants A_m and B_m only.

$$Y(y_o) = A_m\{ \sin(\alpha_m y_o) + J \sinh(\beta_m y_o) \} + B_m\{ \cos(\alpha_m y_o) + (\alpha_m/\beta_m)^2 \cosh(\beta_m y_o) \} \quad (392)$$

As before, equation (392) must be differentiated three times. The first derivative of $Y(y_o)$ with respect to y_o is:

$$Y'(y_o) = A_m\{ \alpha_m \cos(\alpha_m y_o) + J \beta_m \cosh(\beta_m y_o) \} - B_m\{ \alpha_m \sin(\alpha_m y_o) - (\alpha_m^2/\beta_m) \sinh(\beta_m y_o) \} \quad (393)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = -A_m\{\alpha_m^2 \sin(\alpha_m y_o) - J\,\beta_m^2 \sinh(\beta_m y_o)\}$$
$$- B_m\{\alpha_m^2 \cos(\alpha_m y_o) - \alpha_m^2 \cosh(\beta_m y_o)\} \qquad (394)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = -A_m\{\alpha_m^3 \cos(\alpha_m y_o) - J\,\beta_m^3 \cosh(\beta_m y_o)\}$$
$$+ B_m\{\alpha_m^3 \sin(\alpha_m y_o) + \alpha_m^2\beta_m \sinh(\beta_m y_o)\} \qquad (395)$$

Substitution of equations (393), (394), and (395) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sin(\alpha_m b_o)\ \{-\alpha_m^2\} \\ +\sinh(\beta_m b_o)\ \{J\beta_m^2\} \end{array} & \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^2\} \\ +\cosh(\beta_m b_o)\ \{\alpha_m^2\} \end{array} \\ \hline \begin{array}{l} \cos(\alpha_m b_o)\ \{-\alpha_m^3 \\ \quad + K_{y_o}(\pi/a_o)^2 \alpha_m\} \\ +\cosh(\beta_m b_o)\ \{J\beta_m^3 \\ \quad + K_{y_o}(\pi/a_o)^2 J\beta_m\} \end{array} & \begin{array}{l} \sin(\alpha_m b_o)\ \{\alpha_m^3 \\ \quad - K_{y_o}(\pi/a_o)^2 \alpha_m\} \\ +\sinh(\beta_m b_o)\ \{\alpha_m^2\beta_m \\ \quad + K_{y_o}(\pi/a_o)^2\ \alpha_m^2/\beta_m\} \end{array} \end{vmatrix} = 0 \qquad (396)$$

Expansion of the determinant, division by the common multiple α_m, and multiplication by the quantity b_o^5 gives:

$$\sin(\alpha_m b_o)\ \sinh(\beta_m b_o)\ \{-(\alpha_m b_o)^3 (\beta_m b_o)^2 + J(\alpha_m b_o)^2 (\beta_m b_o)^3$$
$$- K_{y_o} (\pi b_o/a_o)^2 [\ (\alpha_m b_o)^3 + J(\beta_m b_o)^3\ ]\ \}$$
$$+ \cos(\alpha_m b_o)\ \cosh(\beta_m b_o)\ \{\ (\alpha_m b_o)^4 (\beta_m b_o) + J(\alpha_m b_o)(\beta_m b_o)^4$$
$$+ K_{y_o} (\pi b_o/a_o)^2 [\ J(\alpha_m b_o)(\beta_m b_o)^2 - (\alpha_m b_o)^2 (\beta_m b_o)\ ]\ \}$$
$$+ \{\ -(\alpha_m b_o)^4 (\beta_m b_o) - J(\alpha_m b_o)(\beta_m b_o)^4$$
$$+ K_{y_o} (\pi b_o/a_o)^2 [\ (\alpha_m b_o)^2 (\beta_m b_o)$$
$$- J(\alpha_m b_o)(\beta_m b_o)^2\ ]\ \} = 0 \qquad (397)$$

Equation (397) represents the governing equation for the combination of any plate aspect ratio and K_{y_o} less than zero. In other words, the roots of equation (397) yield the smallest values of K_{x_o} for any a_o/b_o and tensile K_{y_o}. Consideration of results generated by equation (397) is postponed until the last of the three subcases is presented.

K_{y_o} Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, K_{y_o} can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Moreover, equations (172), (173), and (174) sequentially illustrate that $Y(y_o)$ must take the form shown in equation (175). Note also that equations (163) and (176) define the variables in equation (175).

The first derivative with respect to y_o of this function $Y(y_o)$ in equation (175) is:

$$Y'(y_o) = A_m\theta_m \cosh(\theta_m y_o) + B_m\theta_m \sinh(\theta_m y_o) + C_m\beta_m \cosh(\beta_m y_o) + D_m\beta_m \sinh(\beta_m y_o) \qquad (398)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m\theta_m^2 \sinh(\theta_m y_o) + B_m\theta_m^2 \cosh(\theta_m y_o) + C_m\beta_m^2 \sinh(\beta_m y_o) + D_m\beta_m^2 \cosh(\beta_m y_o) \qquad (399)$$

The third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\theta_m^3 \cosh(\theta_m y_o) + B_m\theta_m^3 \sinh(\theta_m y_o) + C_m\beta_m^3 \cosh(\beta_m y_o) + D_m\beta_m^3 \sinh(\beta_m y_o) \qquad (400)$$

Apply equation (327) to equation (399). this combination fixes D_m in terms of B_m such that $D_m = -(\theta_m/\beta_m)^2 B_m$ Next, couple equations (398) and (400) in such a fashion so that they adhere to the boundary condition expressed in equation (328). C_m is thus related to A_m in the following manner:

$$C_m = -A_m\{ \theta_m^3 + K_{y_o}(\pi/a_o)^2 \theta_m \} / \{ \beta_m^3 + K_{y_o}(\pi/a_o)^2 \beta_m \} \qquad (401)$$

Define the following variable:

$$W = -\{ \theta_m^3 + K_{y_o}(\pi/a_o)^2 \theta_m \} / \{ \beta_m^3 + K_{y_o}(\pi/a_o)^2 \beta_m \} \qquad (402)$$

With the variable W so defined, the $Y(y_o)$ function, equation (175), can be written in terms of the constants A_m and B_m only.

$$Y(y_o) = A_m\{ \sinh(\theta_m y_o) + W \sinh(\beta_m y_o) \} + B_m\{ \cosh(\theta_m y_o) - [\theta_m/\beta_m]^2 \cosh(\beta_m y_o) \} \qquad (403)$$

Three derivatives must again be taken of this $Y(y_o)$ function. The first derivative of $Y(y_o)$ with respect to y_o is:

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COMPOSITE RECTANGULAR PLATES(U) AIR FORCE INST OF TECH
WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. J P MCFADDEN
UNCLASSIFIED   DEC 84 AFIT/GAE/AA/84D-15   F/G 20/11   NL

END
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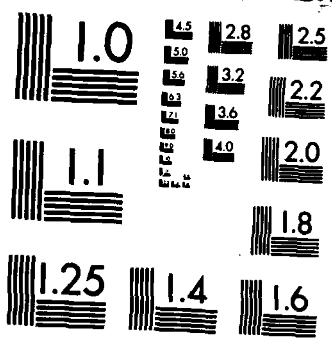
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3
7.1

$$Y'(y_o) = A_m\{ \theta_m \cosh(\theta_m y_o) + W \beta_m \cosh(\beta_m y_o) \}$$
$$+ B_m\{ \theta_m \sinh(\theta_m y_o) - [\theta_m^2/\beta_m] \sinh(\beta_m y_o) \} \quad (404)$$

The second derivative of $Y(y_o)$ with respect to y_o is:

$$Y''(y_o) = A_m\{ \theta_m^2 \sinh(\theta_m y_o) + W \beta_m^2 \sinh(\beta_m y_o) \}$$
$$+ B_m\{ \theta_m^2 \cosh(\theta_m y_o) - \theta_m^2 \cosh(\beta_m y_o) \} \quad (405)$$

Finally, the third derivative of $Y(y_o)$ with respect to y_o is:

$$Y'''(y_o) = A_m\{ \theta_m^3 \cosh(\theta_m y_o) + W \beta_m^3 \cosh(\beta_m y_o) \}$$
$$+ B_m\{ \theta_m^3 \sinh(\theta_m y_o) - \theta_m^2 \beta_m \sinh(\beta_m y_o) \} \quad (406)$$

Substitution of equations (404), (405), and (406) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients A_m and B_m. For non-trivial A_m and B_m, the following determinental equation must hold:

$$\begin{vmatrix} \begin{array}{l} \sinh(\theta_m b_o)\ \{\theta_m^2\} \\ + \sinh(\beta_m b_o)\ \{W \beta_m^2\} \end{array} & \begin{array}{l} \cosh(\theta_m b_o)\ \{\theta_m^2\} \\ + \cosh(\beta_m b_o)\ \{-\theta_m^2\} \end{array} \\ \\ \begin{array}{l} \cosh(\theta_m b_o)\ \{\theta_m^3 \\ \quad + K_{y_o} (\pi/a_o)^2 \theta_m\} \\ + \cosh(\beta_m b_o)\ \{W \beta_m^3 \\ \quad + K_{y_o} (\pi/a_o)^2 W \beta_m\} \end{array} & \begin{array}{l} \sinh(\theta_m b_o)\ \{\theta_m^3 \\ \quad + K_{y_o} (\pi/a_o)^2 \theta_m\} \\ - \sinh(\beta_m b_o)\ \{\theta_m^2 \beta_m \\ \quad + K_{y_o} (\pi/a_o)^2 \theta_m^2/\beta_m\} \end{array} \end{vmatrix} = 0 \quad (407)$$

Expansion of the determinant, division by the common multiple θ_m, and multiplication by the quantity $\beta_m b_o^5$ gives:

$$\sinh(\theta_m b_o)\ \sinh(\beta_m b_o)\ \{-(\theta_m b_o)^3 (\beta_m b_o)^2 + W(\theta_m b_o)^2 (\beta_m b_o)^3$$
$$+ K_{y_o} (\Pi b_o/a_o)^2 [\ W(\beta_m b_o)^3 - (\theta_m b_o)^3]\ \}$$
$$+ \cosh(\theta_m b_o)\ \cosh(\beta_m b_o)\ \{-W(\theta_m b_o)(\beta_m b_o)^4 + (\theta_m b_o)^4 (\beta_m b_o)$$
$$+ K_{y_o} (\Pi b_o/a_o)^2 [\ (\theta_m b_o)^2 (\beta_m b_o) - W(\theta_m b_o)(\beta_m b_o)^2]\ \}$$
$$+ \{-(\theta_m b_o)^4 (\beta_m b_o) + W(\theta_m b_o)(\beta_m b_o)^4$$
$$+ K_{y_o} (\Pi b_o/a_o)^2 [\ W(\theta_m b_o)(\beta_m b_o)^2$$
$$- (\theta_m b_o)^2 (\beta_m b_o)\ ]\ \} = 0 \qquad (408)$$

Attempts at solution of equation (408) for any combination of a_o/b_o and K_{y_o} do not yield the smallest values of K_{x_o}. As a result, further considerations of this equation and this subcase as a whole are dropped.

Discussion of Results

Table XVII gives selected a_o/b_o , K_{y_o}, and K_{x_o} ordered triplets as determined by equation (350). Note that the results generated by equation (350) are given--and indeed are only valid--for positive, or compressive, and zero K_{y_o} . These numbers and plots based upon this data reveal three very important aspects of the buckling characteristics of a laminate supported by this relatively weak set of boundary conditions. First, all values of minimum K_{x_o} for any combination of a_o/b_o and compressive or zero K_{y_o} are achieved when $m = 1$ is utilized in the terms which compose equation (350). More broadly, K_{x_o} plotted versus a_o/b_o for any constant K_{y_o} greater than or equal to zero results in just one continuous curve. There exist no transition points to another curve.

Second, for $K_{y_o} = 0$ data points for a K_{x_o} versus a_o/b_o sketch lie just and virtually indistinguishably below the boundary curve $\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} = 0$ Since $m = 1$ in all cases for this uniaxial buckling ($K_{y_o} = 0$), this boundary value of x_o-buckling coefficient is given by equation (298). Moreover, as just stated the boundary value quoted in equation (298) is a virtually exact approximation for K_{x_o} for the null value of y_o-buckling coefficient.

Third, curves of K_{x_o} versus a_o/b_o for constant compressive K_{y_o} lie well below the zero ordinate for small and intermediate affine aspect ratios. However, these curves continuously trend upward as a_o/b_o increases. For any constant K_{y_o}, the K_{x_o} versus a_o/b_o plot asymptotically approaches a value of approximately $-1.2\,K_{y_o}$. For example, the following data reflects x_o-buckling coefficients (and corresponding y_o-buckling coefficients) for an affine aspect ratio of 100.0 :

a_o/b_o	K_{y_o}	K_{x_o}
100.0	0.5	-0.6078
100.0	1.0	-1.2158
100.0	1.5	-1.8238
100.0	2.0	-2.4318
100.0	2.5	-3.0398
100.0	3.0	-3.6479

Table XVIII gives selected a_o/b_o, K_{y_o}, and K_{x_o} ordered

triplets as determined by equation (397). Note that the results generated by equation (397) are given and are only valid for negative K_{y_0}. Also included is the integer value value of m which produces this minimum K_{x_0}. Furthermore, each entry point which corresponds to a transition point from the m curve to the (m +1) curve is superscripted with a star (*). Especially be aware that discontinuous curves as opposed to the continuous curves discussed above highlight K_{x_0} versus a_0/b_0 plots for tensile K_{y_0}. The statistics presented in Table XVIII expose two key characteristics of laminates under tension in the y_0-direction. First, the transition values of a_0/b_0 increase as K_{y_0} becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative magnitude of K_{y_0}, K_{x_0} attains a limiting value of zero as a_0/b_0 approaches infinity. For this set of boundary conditions, the rate of approach toward this zero asymptote is decidedly slow. For example, for $K_{y_0} = -3.0$, $K_{x_0} = 0.0382$ at $a_0/b_0 = 100.0$ and $K_{x_0} = 0.0348$ at $a_0/b_0 = 110.0$ Note that this asymptotic value of zero differs from those asymptotes presented for compressive K_{y_0}.

Figure 32 represents a plot of K_{x_0} versus a_0/b_0 for twelve distinct values of K_{y_0}. The lowest curve characterizes $K_{y_0} = 3.0$; whereas, the highest depicts

$K_{y_o} = -5.0$ In ascending order the magnitudes of the y_o-buckling coefficients which correspond to the remaining ten curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0 This graph reinforces the concepts that K_{x_o} for a compressive or zero K_{y_o} is determined by one continuous curve and that K_{x_o} for a tensile K_{y_o} is rendered by the lowest values of an infinite number of intersecting curves. In addition, the merging of the family of curves to the zero asymptote for tensile or zero K_{y_o} and to distinct, relatively evenly spaced asymptotes for compressive K_{y_o} is readily apparent.

Figure 33 plots in three dimensions the same information as Figure 32. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 32; however, the quantitative aspect is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table XIX gives selected coordinates of K_{y_o} and K_{x_o} for three distinct values of a_o/b_o --1.2, 2.6, and 5.4 Figures 34, 35, and 36 represent two-dimensional plots at these constant a_o/b_o slices of 1.2, 2.6, and 5.4, respectively. Because the data for each of the curves are derived for two completely different equations ((350) for K_{y_o} positive or zero; (397) for K_{y_o} negative), a mild variance of slope in the vicinity of $K_{y_o} = 0.0$ is observed for $a_o/b_o = 1.2$ and

slightly larger variances for $a_o/b_o = 2.6$ and $a_o/b_o = 5.4$ These breaks are certainly less severe than those observed in corresponding plots for a laminate simply supported on opposite sides and clamped and free on the remaining edges.

TABLE XVII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Free in the y_o-Direction (for K_{y_o} greater than or equal to zero)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	1.5	-3.4959
1	1.0000	1.5	-2.0476
1	1.4000	1.5	-1.8872
1	2.0000	1.5	-1.8424
1	2.8000	1.5	-1.8304
1	3.6000	1.5	-1.8271
1	4.6000	1.5	-1.8255
1	5.4000	1.5	-1.8250
1	0.8000	2.0	-4.9033
1	1.2000	2.0	-3.2291
1	1.6000	2.0	-2.8233
1	2.0000	2.0	-2.6662
1	2.8000	2.0	-2.5444
1	3.6000	2.0	-2.4982
1	4.6000	2.0	-2.4718
1	5.4000	2.0	-2.4606
1	1.0000	2.5	-5.8317
1	1.4000	2.5	-4.2198
1	1.8000	2.5	-3.6956
1	2.2000	2.5	-3.4598
1	2.6000	2.5	-3.3329
1	3.2000	2.5	-3.2290
1	4.4000	2.5	-3.1378
1	5.4000	2.5	-3.1043

TABLE XVIII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the x_o-Direction and Free in the y_o-Direction (for K_{y_o} less than zero)

m	a_o/b_o	K_{y_o}	K_{x_o}
1	0.6000	-5.0	8.0156
1	0.8613*	-5.0	6.7408
2	1.2000	-5.0	4.1690
2	1.6000	-5.0	2.9906
2	2.0000	-5.0	2.4529
2	2.4728*	-5.0	2.1260
3	3.0000	-5.0	1.6601
3	3.6000	-5.0	1.3588
3	4.2000	-5.0	1.1773
3	4.8897*	-5.0	1.0456
4	5.2000	-5.0	0.9685
4	5.4000	-5.0	0.9257
1	0.6000	-3.0	5.9832
1	1.0865*	-3.0	4.2352
2	1.4000	-3.0	2.9061
2	1.8000	-3.0	2.1146
2	2.2000	-3.0	1.7155
2	2.8000	-3.0	1.4072
2	3.1623*	-3.0	1.3000
3	3.8000	-3.0	1.0248
3	4.6000	-3.0	0.8280
3	5.4000	-3.0	0.7120
1	0.6000	-1.0	3.9026
1	1.0000	-1.0	2.1707
1	1.4000	-1.0	1.7004
1	1.8258*	-1.0	1.5000
2	2.2000	-1.0	1.1276
2	2.6000	-1.0	0.8936
2	3.2000	-1.0	0.6932
2	3.8000	-1.0	0.5800
2	4.6000	-1.0	0.4923
2	5.4000	-1.0	0.4407

TABLE XIX

K_{x_o} Versus K_{y_o} for Various Plate Aspect Ratios for a Laminate Simply Supported in the x_o-Direction and Free in the y_o-Direction

m	a_o/b_o	K_{y_o}	K_{x_o}
2	1.2000	-5.0	4.1690
2	1.2000	-3.0	3.6323
1	1.2000	-2.0	3.0104
1	1.2000	-1.0	1.8768
1	1.2000	0.0	0.6932
1	1.2000	0.5	0.0039
1	1.2000	1.0	-0.8656
1	1.2000	1.5	-1.9356
1	1.2000	2.0	-3.2291
1	1.2000	2.5	-4.7697
3	2.6000	-5.0	1.9872
2	2.6000	-3.0	1.4866
2	2.6000	-1.0	0.8936
1	2.6000	0.0	0.1467
1	2.6000	0.5	-0.4770
1	2.6000	1.0	-1.1367
1	2.6000	1.5	-1.8320
1	2.6000	2.0	-2.5637
1	2.6000	2.5	-3.3329
4	5.4000	-5.0	0.9257
3	5.4000	-3.0	0.7120
2	5.4000	-1.0	0.4407
1	5.4000	0.0	0.0331
1	5.4000	0.5	-0.5776
1	5.4000	1.0	-1.1973
1	5.4000	1.5	-1.8250
1	5.4000	2.0	-2.4606
1	5.4000	2.5	-3.1043

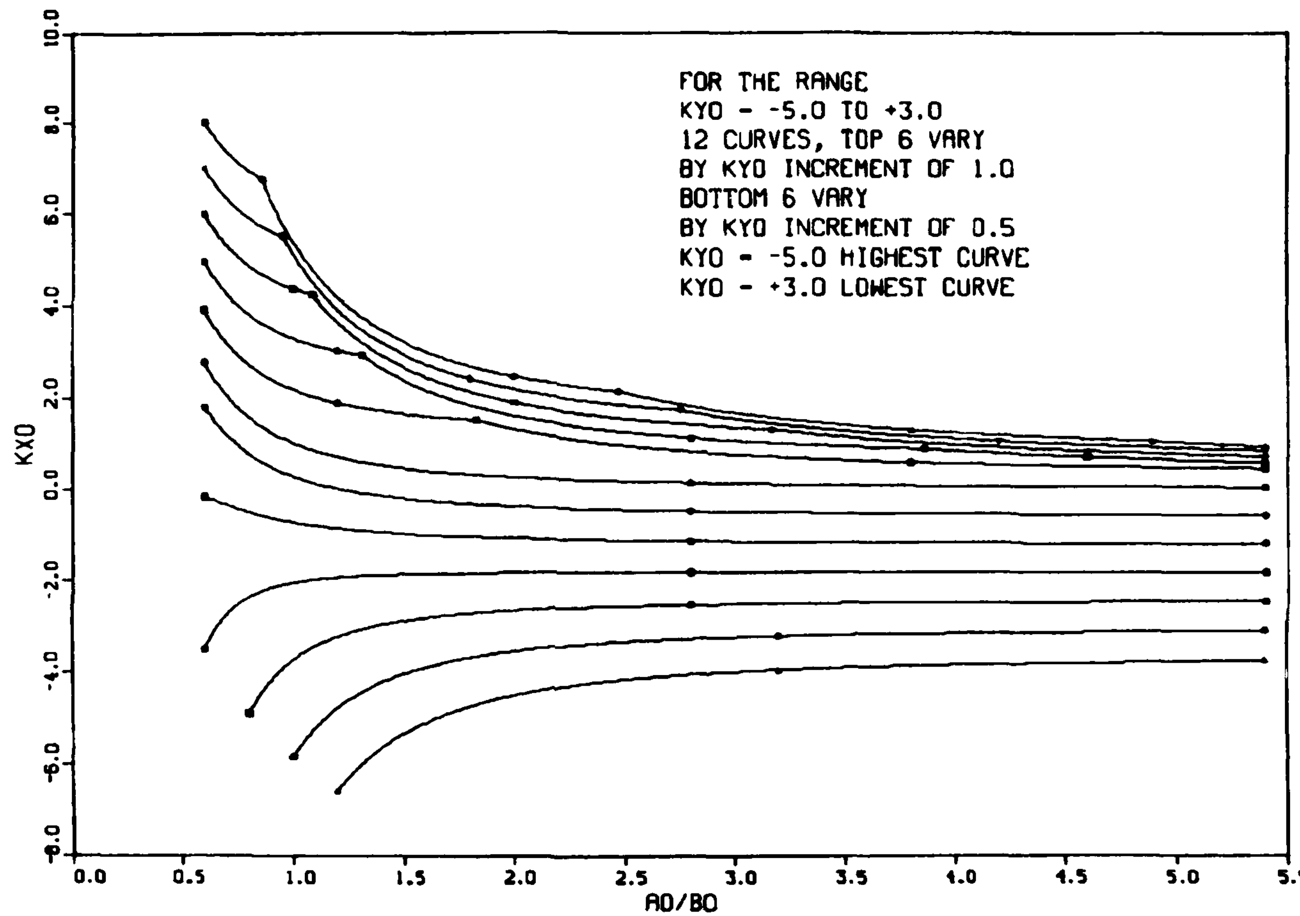

FIGURE 32

XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-F-S-F LAMINATE FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES

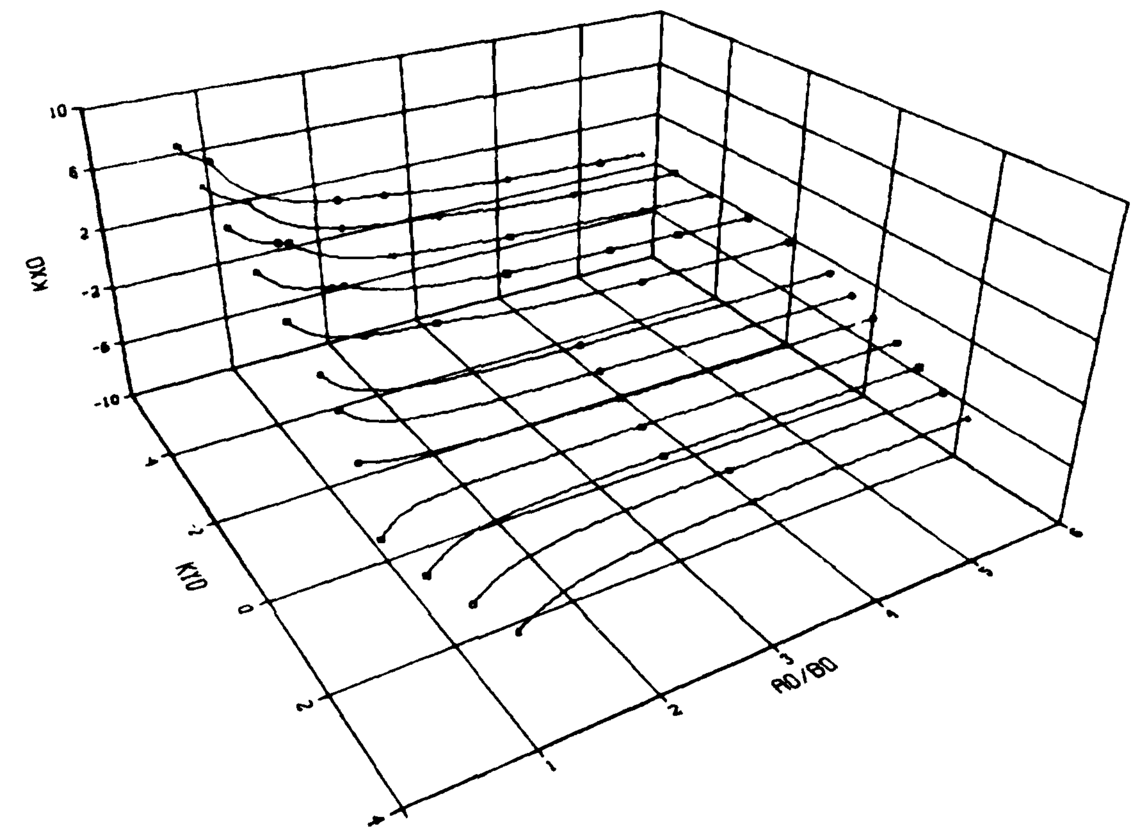

FIGURE 33

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-F-S-F LAMINATE

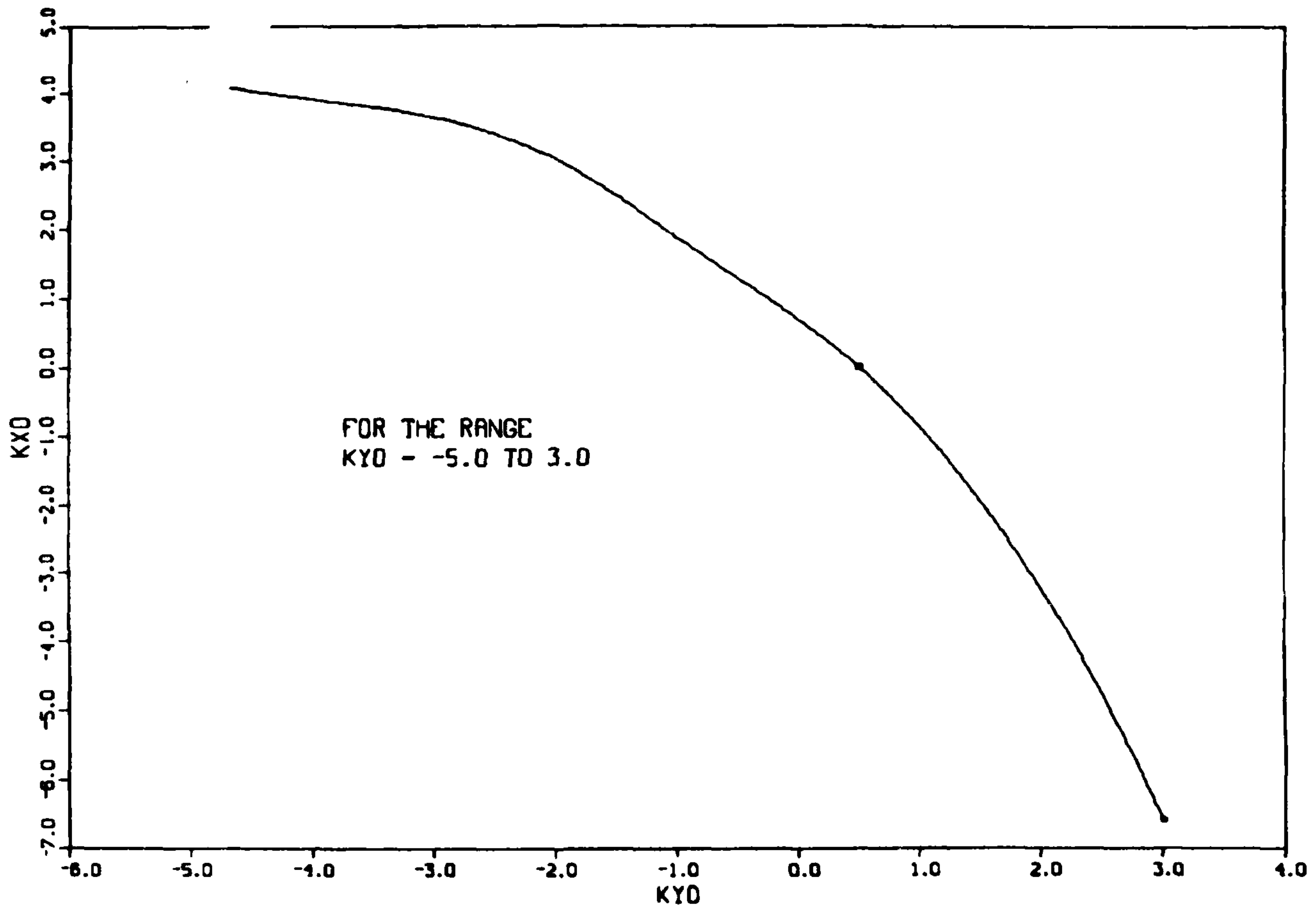

FIGURE 34

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.2 FOR AN S-F-S-F LAMINATE

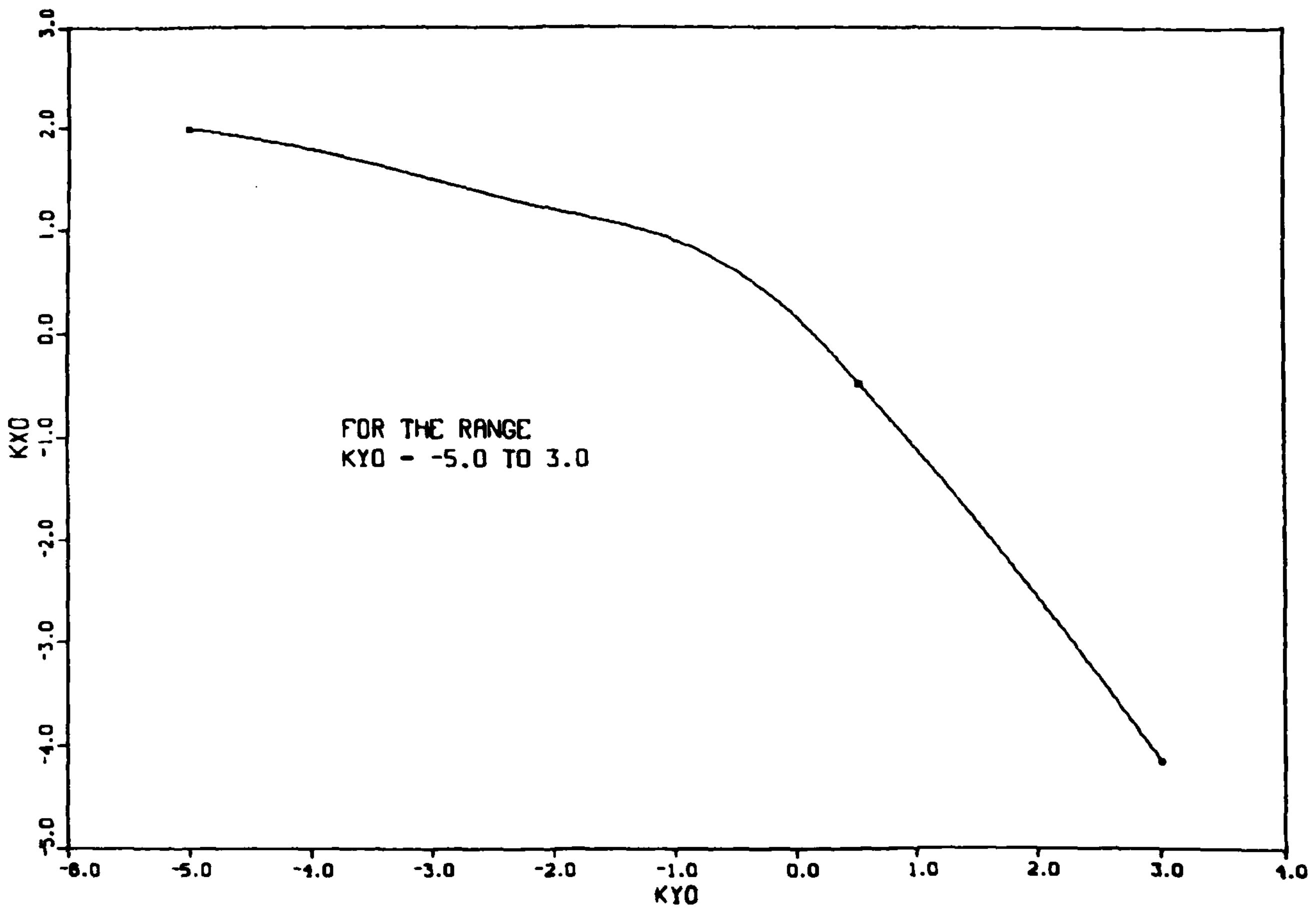

FIGURE 35

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.6 FOR AN S-F-S-F LAMINATE

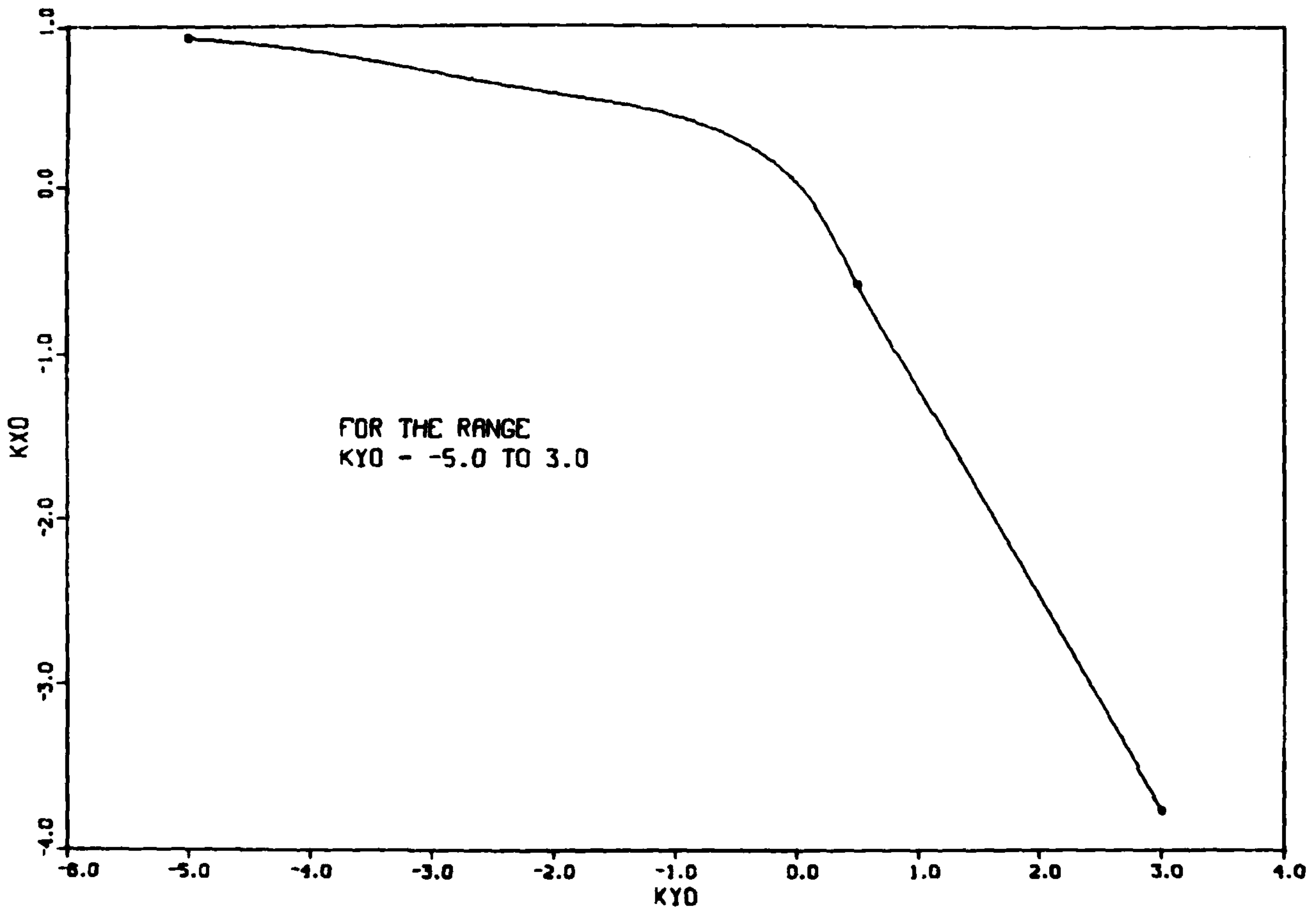

FIGURE 36

XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 5.4 FOR AN S-F-S-F LAMINATE

IX. Conclusion

The plate buckling problem for a specially orthotropic, symmetric composite laminate depends rather weakly on the starred bending stiffness ratio D^*. This ratio of course lies within the range zero to one, and in fact proves to be quite invariant of the actual magnitude within these limits. As a result, the selection of the null value of D^* both facilitates the solution process and returns excellent approximations of plate buckling behavior.

The seven sets of edge constraints examined blanket a wide range of support condition combinations. Any configuration for which at least one set of opposite edges is simply supported has been rigorously examined. Moreover, the clamped on all sides case (probably the most important) is similarly and rather thoroughly presented. It is hoped that this group of boundary conditions constitutes a set large enough so that they may be meaningfully employed in a design process.

Finally, the fact that buckling coefficients and aspect ratio are expressed in affine space allows sweeping generality. The buckling surfaces for each of the seven sets of boundary conditions can be utilized for any composite material. The coordinates need be scaled only by the first

two diagonal elements of the bending stiffness array. From this flexibility one can further infer that these buckling results hold for specially orthotropic, symmetric laminates of today and even for those yet to be devised.

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James P. McFadden was born on 13 September 1960 in East Chicago, Indiana. He graduated from high school in Whiting, Indiana in 1978 and attended the University of Notre Dame from which he received a Bachelor of Science in Civil Engineering in May 1982. Upon graduation, he worked for Aeronautical Systems Division at Wright-Patterson AFB. He entered the School of Engineering, Air Force Institute of Technology, as a part-time student in July 1982.

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The biaxial plate buckling problem for specially orthotropic, symmetric laminates is transformed from Cartesian to doubly affine space. Setting the starred bending stiffness ratio D* (which ranges from zero to one) to the null value enables ready and very accurate solution of the buckling problem. Seven sets of boundary restraint configurations are examined, and corresponding buckling surfaces are presented. The character of these results vary widely between the strongest and weakest sets of support conditions. In order to prevent buckling for the weakest conditions, considerable tension must be provided on parallel edges for just small amounts of compression applied on the opposite set of edges.